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On the Equivalence between Iterated Application of Choice Rules and Common Belief of Applying these Rules*

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Abstract

One central issue tackled in epistemic game theory is whether for a general class of strategic games the solution generated by iterated application of a choice rule gives exactly the strategy profiles that might be realized by players who follow this choice rule and commonly believe they follow this rule. For example, Brandenburger and Dekel (1987) and Tan and Werlang (1988) have established that this coincidence holds for the choice rule of strict undominance in mixtures in the class of finite strategic games, and Mariotti (2003) has established that this coincidence holds for Bernheim's (1984) choice rule of point rationality in the class of strategic games in which the strategy sets are compact Hausdorff and the payoff functions are continuous. In this paper, we aim at studying this coincidence in a general way. We seek to figure out general conditions of the choice rules ensuring it for a general class of strategic games. We state four substantial assumptions on choice rules. If the players' choices rules satisfy - besides the technical assumption of regularity - the properties of reflexivity, monotonicity, Aizerman's property, and the independence of payoff equivalent conditions, then this coincidence applies. This result proves to be strict in the following sense. None of the four substantial properties can be omitted without eliminating the coincidence.

JEL Classification Number: C72, D83.

Keywords: Iterative deletion procedure, common belief, choice rule, epistemic game theory

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1 Introduction

One central issue tackled in epistemic game theory is whether for a general class of strategic games the solution generated by iterated application of choice rules gives exactly the strategy profiles that might be realized by players who follow these choice rules and commonly believe they follow them (i.e., all players believe that all players follow these rules, all players believe that all players believe that all players follow these rules, and so on ad infinitum). If such coincidence holds, the solution concept of iterated application of choice rules is said to be *(epistemically) characterized by choice-rule following behavior and common belief of it*. The most prominent example of a choice rule to which such an epistemic characterization applies is the strict undominance in mixtures. It is known that this rule favors those available strategies that are not strictly dominated by some mixed strategy.[1] As shown by Brandenburger and Dekel (1987) and Tan and Werlang (1988) for the class of finite strategic games (i.e., strategic games in that the players' strategy sets are finite), the strategy profiles surviving the iterated application of this choice rule are exactly the ones that might be chosen if players are Bayesian rational and this is commonly believed among them. Since according to Lemma 3 of Pearce (1984) being Bayesian rational means nothing else than complying with the choice rule of strict undominance in mixtures, it follows that this iterative deletion procedure is characterizable by choice-rule following behavior and common belief of it.

Meanwhile, other prominent choice rules have been identified that share this epistemic characteristic, e.g., the choice rules of point rationality and strict undominance (in pure strategies). The former choice rule has been put forward by Bernheim (1984). It favors any available strategy yielding the highest payoff for at least one strategy combination chosen by the player's opponents. This rule underlies Bernheim's (1984) well-known game-theoretic solution concept of point rationalizability, which gives all strategy profiles surviving the iterated application of this rule. Mariotti (2003) proved that for the class of strategic games in which the strategy sets are compact Hausdorff and the payoff functions are continuous point rationalizability is characterizable by choice-rule following behavior and common belief of it. The choice rule of strict undominance favors any available strategy that is not strictly dominated by some other available strategy. As can be easily inferred from Theorems 2 and 4 of Chen et al. (2007), the above epistemic characteristic is also inherent in this choice rule whenever the same class of strategic games is considered as in Mariotti (2003).[2]

Nevertheless, there are some prominent choice rules that do not imply such characterization, even if only the class of finite strategic games is considered. We adduce as an instance the choice

[1]As usual, a strategy is said to be strictly dominated by some other strategy whenever the payoff of the former is less than the payoff of the latter, whatever the player's opponents decide to do. A mixed strategy represents a randomization on the available strategies. A strategy is said to be strictly dominated by some mixed strategy whenever the payoff of the strategy is less than the expected payoff of the mixed strategy, whatever the player's opponents decide to do. We refer to Appendix C of this paper for a formal definition of these choice rules.

[2]The focus of Chen et al. (2007) is on an alternative iterative deletion procedure that they call $IESDS^*$. Unlike the standard procedure of the iterated deletion of strictly dominated strategies, their deletion procedure allows for transfinite recursion and for deleting strategies strictly dominated by strategies that have previously been deleted. However, as established in their Theorem 2, their procedure coincides with the standard procedure if the strategy sets are compact Hausdorff and the payoff functions are continuous.

rule of maximin, which has been proposed by Wald (1945) and ranks the available strategies according to their security levels (i.e., their worst possible payoffs). Under this rule, the available strategies having the highest security level are considered favorable. As is demonstrated next, there exist finite strategic games in which the set of strategy profiles surviving the iterated application of the maximin rule differs from the set of strategy profiles that might be chosen if all players adhere to this choice rule and this is commonly believed among them. To establish this claim, consider strategic game Γ_1 depicted in Figure 1.

		Player C	
		l	r
Player R	u	$(1,1)$	$(2,2)$
	d	$(0,1)$	$(1,0)$

Figure 1: Strategic game Γ_1

To begin with, strategic game Γ_1 is solved by iterated application of the maximin rule. Note that, in our paper, the iterated application of a choice rule is defined as a *maximal iterative deletion procedure*. For such procedures, it is required that in each round of deletion any strategy of each player be eliminated that proves to be unfavorable according to this choice rule. Regarding the iterated application of the maximin rule on strategic game Γ_1, it turns out that strategy d of player R and strategy r of player C must be deleted in the first round. Hence, strategy profile (u, l) becomes the unique solution of this procedure. In the following, we compare this solution with the set of strategy profiles that might be realized if both players comply with the maximin rule and commonly believe this (i.e, both players believe that both players comply with this rule, both players believe that both players believe that both players comply with this rule, and so on ad infinitum). This analysis is split in two issues. First, we ask, whether the strategy profile (u, l) (i.e., the solution of the iterated application of the maximin rule) might be chosen by such players. Afterwards, we examine whether strategy profiles exist that might be chosen by such players but do not survive the iterated application of the maximin rule.

Let us start with the first issue and suppose that strategy profile (u, l) is chosen, that both players follow the maximin rule, and that there is common belief among them that they follow this rule. Because C is assumed to act according to the maximin rule, playing l can only result if he considers it possible that R chooses d (otherwise, he would obviously choose r). Note that, regardless of which strategies of player C player R considers possible, R never chooses d if she complies with the maximin rule. On the other hand, our assumption of common belief of maximin rule following behavior implies that C believes R follows this rule. Hence, C's attitude of considering it possible that R will choose strategy d is at odds with his belief R follows the maximin rule. This contradiction reveals that strategy profile (u, l), which has been obtained by iterated application of the maximin rule, is incompatible with the assumption that both players follow this rule and this is commonly believed.

Let us turn to the second issue and examine if strategic game Γ_1 contains some strategy profile that might be realized by players who follow the maximin rule and commonly believe this. As just shown, strategy profile (u, l) is incompatible with this assumption. This also holds for strategy profiles (d, l) and (d, r) since choosing d is never favorable under the maximin rule. It remains to check strategy profile (u, r). In order to model the players' beliefs about the choices and the other players' beliefs, we resort to an idea of Harsanyi (1967/68) and represent them by types. For our purpose, it will be sufficient to consider the state of a world in which player R chooses u and is of type t_R, and player C chooses r and is of type t_C, and where these types are specified as follows. Type t_R represents R's belief that C chooses r and is of type t_C, whereas type t_C represents C's belief that R chooses u and is of type t_R. Obviously, it follows from these assumptions (and the assumption implicit in type space models that all players know their own choices and beliefs) that in this state (*i*) both players believe that R chooses u and C chooses r, that (*ii*) both players believe (*i*), that *(iii)* both players believe *(ii)*, and so on ad infinitum. By statement (*i*), it is justified to conclude that both players act in this state as if they follow the maximin rule. Consequently, we can infer from statement (*ii*) that, in this state, they also believe they follow the maximin rule and from statement (*iii*) that, in this state, they also believe that they believe they follow this rule. Since we can proceed in this way ad infinitum, we come to the following conclusion. In this state, strategy profile (u, l) is realized, both players act according to the maximin rule, and there is common belief among them that they act according to the maximin rule. Although this strategy profile does not survive the iterated application of the maximin rule, it might be realized by players applying this rule and commonly believe they apply this rule.

As just exemplified by the maximin rule, the solution originating from the iterated application of a choice rule on a finite strategic game might differ from that realized by players who follow this rule and commonly believe this. It is noteworthy that this divergence emerges under many prominent choice rules besides that of maximin such as the choice rule of weak undominance, that of weak undominance in mixtures, and that of minimax regret.[3] The latter has been forcefully put forward by Niehans (1948) and Savage (1951), and the solution concept of iterated regret minimization originating from the iterated application of this choice rule has recently gained attention among game theorists (see, e.g., Halpern and Pass, 2012).

In this paper, we aim at solving the issue of the coincidence between the solution concept of iterated application of choices rules and the solution concept of choice-rule following behavior and common belief of it in a general way. We seek to figure out general conditions of the choice rules ensuring this coincidence for a general class of strategic games. This issue is not new. Articles aiming to detect such conditions have already been published. To the best of our knowledge, there

[3]As usual, a strategy is said to be weakly dominated by some other strategy whenever (*i*) the payoff of the former does not exceed the payoff of the latter, whatever the player's opponents decide to do, and (*ii*) the payoff of the former is less than the payoff of the latter for some decision of the player's opponents. A strategy is said to be weakly dominated by some mixed strategy whenever (*i*) the payoff of the strategy does not exceed the expected payoff of the mixed strategy, whatever the player's opponents decide to do and (*ii*) the payoff of the strategy is less than the payoff of the mixed strategy for some decision of the player's opponents. While the choice rule of weak undominance favors the strategies that are not weakly dominated by some other strategy, the choice rule of weak undominance in mixtures favors only the strategies that are not weakly dominated by some mixed strategy. The choice rule of minimax regret ranks the strategies according to their worst possible regrets. The regret of a strategy at some strategy choice of the player's opponents is defined as a deviation of the payoff of this strategy from the highest possible payoff attainable by some available strategy with this strategy choice of the opponents. Finally, this rule selects the strategies whose worst possible regret is the lowest among all available strategies. A formal definition of these choice rules is provided in Appendix C.

are two papers that have taking up this issue. However, both have limitations, which we address to overcome.

The first article on this issue was published by Epstein (1997). In it, a preference-based approach is pursued. Its starting point is a preference model that assigns a set of possible preference relations to each player. Based on this model, preference-based choice rules can be specified. A player is said to follow a choice rule of preference-rationalizability if her chosen strategy belongs to the best strategies according to some preference relation of this set of possible preference relations. The article's main achievement is that general assumptions on the preference model are identified so that the solution obtained by the iterated application of choice rules of preference-rationalizability is characterizable by behavior according to this choice rule and common belief of it.[4] However, we would like to point to two limitations of this result. First, as already outlined, the basic notion of this result is that of a preference relation from which a choice rule is constructed. Therefore, choice rules that could not be rationalized by some set of preference relations are not considered in the analysis of Epstein (1997).[5] Second, the above result has been proved only for finite strategic games.

The second article is by Apt and Zvesper (2010). Unlike Epstein (1997), they start directly with the notion of a choice rule and put no restrictions on the cardinality of the players' strategy sets.[6] They show that if the choice rules satisfies their property of monotonicity, then the iterated application of such choice rules proves to be characterizable by choice-rule following behavior and common belief of it. However, their construction of the process of iterated application of choice rules differs from that in standard game theory with respect to two aspects. First, while standard game theory defines this process as a finite recursion, Apt and Zvesper (2010) make use of transfinite ordinals. Second, while in iterative deletion procedures of standard game theory the players' choices rules are applied on the remaining strategies in each round of the procedure, in their iterative deletion procedure the players' choice rules are applied on the initial strategies in each round of the process.

The task of this paper is to overcome the limitations of these two articles. Like Apt and Zvesper (2010), our epistemic analysis starts with the notion of a choice rule and deals with strategic games with arbitrary strategy sets. However, unlike them, we define the process of iterated applications of choice rules in the standard way. In order to require only the use of finite ordinals, we endow strategic games with a topological structure. Indeed, for this purpose we assume that the strategy sets are compact Hausdorff and the players' payoff functions are continuous as in Mariotti's (2003) analysis. Henceforth, such strategic games are referred to as regular strategic games.

To accomplish the aforementioned task, we proceed as follows. In the succeeding section, we explain how strategic games are decomposed into decision problems under uncertainty. Furthermore, we introduce the notion of a choice rule, which is our basic device to solve these decision

[4]We remark that Epstein (1997) does not use the term choice rule is never used. Rather, Epstein calls a player $\mathscr{P}_i^*$-rational whenever she chooses a strategy that belongs to the best strategies according to some preference relation of set $\mathscr{P}_i^*$ of possible preference relations. However, to remain consistent with our other statements, and without changing the content, we have rephrased Epstein's result in our terminology.

[5]In Section 2, we present a trivial example of a choice rule that cannot be induced by any preference model.

[6]Moreover, Apt and Zvesper (2010) do not use the term choice rule. Rather, they introduce the notion of an optimality property. It can be easily shown, however, that this notion corresponds with our notion of a choice rule.

problems. Section 3 deals with the solution concept $\text{IA}_{\mathcal{C}}$ of iterated application of choice rules $\mathcal{C}$. Properties of choice rules are identified so that the solution concept $\text{IA}_{\mathcal{C}}$ is non-empty and stable. In this paper, stability refers to two aspects. First, it requires that the solution generated by iterated application of choice rules be irreducible with respect to further applications of the choice rules (or to put it differently, the use of finite ordinal numbers suffices to find the solution of the iterated application of choice rules). Second, it requires that the solution have the best choice property, which is a generalization of Pearce's (1984) best response property. This demands that every strategy of every player surviving the iterated application of choice rules must be favorable in the decision problem, where (*i*) the player can choose among all her initial strategies but (*ii*) considers possible only the strategies of her opponents that survive the process of iterated application of choice rules. In Section 4, we introduce the solution concept of $\text{CB}_{\mathcal{C}}$, which gives the strategy profiles that might be realized if players follow choice rules $\mathcal{C}$ and this is commonly believed. The construction of this solution concept differs essentially from that of $\text{IA}_{\mathcal{C}}$. As described in detail in Section 3, the latter concept follows the standard approach of game theory (we also refer to it as the ad hoc approach) and is based only on the objective items of the strategic games (i.e., those listed in the forthcoming definition of a strategic game, see Definition 2.1). In contrast, the solution concept of $\text{CB}_{\mathcal{C}}$ is constructed on a broader basis. It is based not only on the objective items of the strategic game but also on its subjective items like the reasoning of the players about the choices and the reasoning of their opponents. To capture these items, we resort to an idea of Harsanyi (1967/68) and supplement the strategic game with a type space. However, unlike standard epistemic game theory, in which the players' types are endowed with probability measures on the opponents' strategies and types, our type space model is of a qualitative nature. In this paper, each type of each player is associated with a (closed) possibility set on the opponents' strategy-type combinations. Such type space models are known as possibility structures in epistemic game theory (cf. the terminology in the survey of Siniscalchi, 2008). The main result of this paper is presented in Section 5. There we list properties of choice rules ensuring that the solution of $\text{IA}_{\mathcal{C}}$ coincides with the solution of $\text{CB}_{\mathcal{C}}$. In Appendix A, it is established that this result is weak in the sense that none of these properties can be canceled without eliminating this coincidence. The proofs of our remarks are relegated to Appendix B. The formal specifications of the choice rules described verbally in the Introduction and in the main text are provided in Appendix C.

2 Decomposing Strategic Games into Decision Problems

As outlined in the Introduction, this paper compares two solution concepts that are the result of different perspectives on games. Despite their divergent geneses, their constructions share common characteristics. For both solution concepts, games are decomposed into decision problems to which choice rules are applied. The objective of this section is to explain how this decomposition is accomplished. For this purpose, we provide first the definitions of the three basic concepts of this paper. These are strategic game, decision problem and choice rule. Subsequently, we link them with each other.

Definition 2.1 *A strategic game is a tuple* $\Gamma := (I, (S^i, z^i)_{i \in I})$ *where*

- *I denotes a non-empty, finite set of players,*
- *S^i denotes the topological strategy space of player i*
- *$z^i : S \to \mathbb{R}$ denotes the payoff function of player i where $S := \prod_{j \in I} S^j$ is endowed with the product topology.*

A strategic game is called regular if, for each player $i \in I$, strategy space S^i is compact Hausdorff and payoff function z^i is continuous.

Consider a group $J \subseteq I$ of players who are taking part in strategic game $\Gamma := (I, (S^i, z^i)_{i \in I})$. The product set $S^J := \prod_{j \in J} S^i$ consisting of all strategy combinations of the players belonging to J is assumed to be endowed with the product topology. Obviously, if strategic game Γ is regular, then product space S^J is compact Hausdorff. As usual, we write S and S^{-i} instead of S^I and $S^{I \setminus \{i\}}$, respectively.

The product set $R := \prod_{i \in N} R^i$ where $R^i \subseteq S^i$ holds for any $i \in I$ is termed a *restriction of strategic game* Γ. We denote the set of all restrictions of strategic game Γ by $\mathscr{R}_\Gamma$. Let $R \in \mathscr{R}_\Gamma$ be a restriction of Γ, then R^i and $R^{-i} := \prod_{j \in I \setminus \{i\}} R^j$ are called *restriction of player i* and *restriction of i's opponents*, respectively. A restriction $R \in \mathscr{R}_\Gamma$ being a closed (compact) subset of S is referred to as a *closed* (resp. *compact*) restriction of Γ. A strategy s^i is said to belong to restriction R whenever $s^i \in R^i$ holds. Strategic game $\Gamma|_R = (I, (R^i, z^i|_{R^i})_{i \in I})$ where $R \in \mathscr{R}_\Gamma$ holds and $z^i|_R$ denotes the restriction of i's payoff function z^i on domain R is called *reduction of game* Γ *on restriction* R.

Let $i \in I$ be some player of game Γ. Her payoff function z^i specifies the payoff $z^i(s)$ (usually interpreted as monetary payoff) she receives if strategy profile $s \in S$ is realized. The set $\mathbb{R}$ of real numbers is assumed to be endowed with the standard (i.e., Euclidean) topology. If the strategic game is regular, the Closed Map Lemma implies that payoff function z^i is closed.[7] Given some strategy $s^i \in S^i$, then $z^i(s^i; .) : S^{-i} \to \mathbb{R}$ denotes the mapping assigning payoff $z^i(s^i, s^{-i})$ to each strategy combination $s^{-i} \in S^{-i}$. Given some strategy $s^{-i} \in S^{-i}$, then $z^i(.; s^{-i}) : S^i \to \mathbb{R}$ denotes the mapping assigning payoff $z^i(s^i, s^{-i})$ to each strategy $s^i \in S^i$. Obviously, if payoff function z^i is continuous, mappings $z^i(s^i; .)$ and $z^i(.; s^{-i})$ are continuous too. Moreover, both mappings are also closed whenever a regular strategic game is presupposed.

A strategic game is termed *finite* whenever the strategy space of every player is finite. Without difficulty, it can be proved that finite strategic games are regular if and only if the strategy space of every player is endowed with the discrete topology (i.e., every subset of the strategy space is an open set). Unlike our Definition 2.1, in game theory textbooks strategic games are usually introduced without any topological assumptions. However, in case of finite strategic games the regularity premise proves to be non-restrictive. Indeed, if the finite strategy spaces of the players are supplemented with the discrete topology, they become compact Hausdorff and the payoff functions become continuous. Hence, our regularity assumption is innocuous for strategic games with finite strategy spaces.

As aforementioned, the two solution concepts that we analyze in the succeeding sections have components in common. Both constructions are based on the decomposition of games into de-

[7] A mapping $f : X \to Y$ from topological space X on topological space Y is called closed if $f(A) := \{f(x) : x \in A\}$ is closed in Y for any set A closed in X. The Closed Map Lemma says that continuous functions whose domains are compact and codomain Hausdorff are closed.

cision problems. In this paper, a *decision problem* is defined by tuple $\Phi := (P, \mathfrak{A})_\Theta$ and consists of the three attributes condition space Θ, possibility set P, and constraint $\mathfrak{A}$. These attributes are interpreted as follows.

Condition space Θ is a topological space that encompasses all conceivable conditions (or, synonymously, circumstances) the decision maker might take into account in her decision making process. Such space is called *regular* whenever it is compact Hausdorff. In general, we denote condition spaces by Greek capital letters (e.g., by Θ) and their elements by Greek lower-case letters (e.g., θ). For varied reasons, which are detailed in the succeeding sections, some of the conceivable circumstances might be precluded by the decision maker. *Possibility set* $P \subseteq \Theta$ is the set that comprises all circumstances not precluded by her. A condition belonging to possibility set P is said to be *considered possible by the decision maker*.

Constraint $\mathfrak{A}$ is the set of alternatives available for the decision maker. Alternatives are represented by *payoff profiles on condition space* Θ. Such profiles are mappings assigning a real number (i.e., a monetary payoff) to each conceivable condition. In Microeconomic Theory literature, payoff profiles are also known as condition contingent monetary payoffs. The set of all payoff profiles on Θ is denoted by $\mathbb{R}^\Omega$. Specific subsets of $\mathbb{R}^\Omega$ are represented by specific characters. The set of all bounded payoff profiles on Θ is denoted by $\mathbf{B}(\Theta)$ and set of all continuous payoff profiles on Θ is denoted by $\mathbf{C}(\Theta)$. Finally, the set of all bounded and continuous payoff profiles on Θ is represented by $\mathbf{BC}(\Theta)$. Throughout the paper, we suppose that $\mathbf{B}(\Theta)$ is endowed with the topology induced by the sup norm $||.||_\infty$. Usually, constraints are denoted by Fraktur capital letters (e.g., $\mathfrak{A}$, $\tilde{\mathfrak{A}}$, ...) and payoff profiles by Fraktur lower-case letters (e.g., $\mathfrak{a}$, $\mathfrak{b}$, ...). The θth component of payoff profile $\mathfrak{a}$ is denoted by $\mathfrak{a}_\theta$ and indicates the payoff received by the decision maker whenever she has chosen payoff profile $\mathfrak{a}$ and circumstance θ occurs. Avoiding notational overload, we frequently make use of the following notational rule of simplification. If the condition space Θ is unequivocally fixed, decision problem $(P, \mathfrak{A})_\Theta$ is expressed in the abbreviated form $(P, \mathfrak{A})$.

The system of all decision problems under basic condition space Θ is denoted by $\mathscr{D}_\Theta$. We refer to $\mathscr{D}_\Theta$ as the *complete system of decision problems under condition space* Θ. In the following, we consider frequently subsystems $\tilde{\mathscr{D}}_\Theta \subseteq \mathscr{D}_\Theta$ of decision problems. In particular, we pay attention to those decision problems Φ of system $\mathscr{D}_\Theta$ that satisfy the following two properties. First, their possibility sets must be closed in Θ and, second, their constraints must be compact subsets of $\mathbf{BC}(\Theta)$. Such decision problems are termed *regular decision problems* by us.

A *choice rule* is a device applied by a decision maker to solve decision problems. Formally, a choice rule $\mathcal{C}$ is a mapping that assigns to each decision problem $\Phi := (P, \mathfrak{A})_\Theta \in \mathscr{D}_\Theta$ of every condition space Θ a (possibly empty) set $\mathcal{C}(\Phi) \subseteq \mathfrak{A}$ of payoff profiles. The set $\mathcal{C}(\Phi)$ is called the *choice set* for decision problem Φ under choice rule $\mathcal{C}$, and the payoff profiles contained in this set are called *favorable* (or *best*) payoff profiles in decision problem Φ under choice rule $\mathcal{C}$. Available payoff profiles that do not belong to the choice set are called unfavorable. To maintain clarity, we simply write $\mathcal{C}(P, \mathfrak{A})_\Theta$ instead of $\mathcal{C}\left((P, \mathfrak{A})_\Theta\right)$.

In Table C.1 of Appendix C several choice rules are specified. This list contains all choice rules mentioned in the Introduction and main text of this paper. Most of them are well-known and often applied in decision theory and game theory. In this section, we focus mainly on the choice rules of strict undominance (in pure payoff profiles) $\mathcal{SU}_p$, strict undominance in mixed payoff profiles

$\mathcal{SU}_m$, point rationality $\mathcal{PR}$, strict dominance $\mathcal{SD}$, modified strict dominance $\mathcal{SD}_+$ and maximin $\mathcal{MM}$. They are adduced to clarify our concepts.

In accordance with the explanations in the Introduction, the choice rule of strict undominance of pure payoff favors the available payoff profiles for that no other available payoff profile exists that yields a higher payoff in every condition considered possible by the decision maker. The choice rule of strict undominance in mixed payoff profiles selects the available payoff profiles for that no mixture of available payoff profiles exists that yields a higher payoff in every condition considered possible. An available payoff profile is favorable under the choice rule of point rationality whenever, at some condition considered possible, it yields the highest possible payoff among available payoff profiles. The choice rule of strict dominance favors the available payoff profiles whose payoffs exceed the payoffs of any other available payoff profile in every condition considered possible. Without difficulty, we recognize that

$$\mathcal{SD}(P,\mathfrak{A})_\Theta \subseteq \mathcal{PR}(P,\mathfrak{A})_\Theta \subseteq \mathcal{SU}_m(P,\mathfrak{A})_\Theta \subseteq \mathcal{SU}_p(P,\mathfrak{A})_\Theta$$

is satisfied for every decision problem $(P,\mathfrak{A})_\Theta \in \mathscr{D}_\Theta$ of any condition space Θ. As its name suggests, the choice rule of modified strict dominance rule is a variant of the strict dominance rule. It corresponds with the strict dominance rule if the latter has a solution, and, otherwise, it views as favorable every available payoff profile. The maximin rule selects the available payoff profiles whose security level (i.e., the greatest payoff that is ensured at every condition considered possible) is higher or equal than the security level of any other available payoff profile. The following definition lists technical properties of choice rules that will be crucial for solving the issues of existence, closedness and stability addressed in the succeeding section.

Definition 2.2 *Let Θ be some condition space and $\tilde{\mathscr{D}}_\Theta \subseteq \mathscr{D}_\Theta$ be some system of decision problems. A choice rule $\mathcal{C}$ is called*

- *non-empty in $\tilde{\mathscr{D}}_\Theta$ if $\mathcal{C}(\Phi)$ is non-empty for any regular decision problem $\Phi \in \tilde{\mathscr{D}}_\Theta$ with non-empty possibility set and non-empty constraint.*
- *closed in $\tilde{\mathscr{D}}_\Theta$ if $\mathcal{C}(\Phi)$ is closed in $\mathbf{B}(\Theta)$ for any regular decision problem $\Phi \in \tilde{\mathscr{D}}_\Theta$.*
- *possibility set continuous from above in $\tilde{\mathscr{D}}_\Theta$ if*

$$\bigcap_{k\in K} \mathcal{C}(P_k,\mathfrak{A}) \subseteq \mathcal{C}(P,\mathfrak{A})$$

  is satisfied for any net $(P_k,\mathfrak{A})_{k\in K}$ of regular decision problems of $\tilde{\mathscr{D}}_\Theta$ and for any regular decision problem $(P,\mathfrak{A}) \in \tilde{\mathscr{D}}_\Theta$ where $P_k \searrow P$ holds.[8]
- *constraint continuous from above in $\tilde{\mathscr{D}}_\Theta$ if*

$$\bigcap_{k\in K} \mathcal{C}(P,\mathfrak{A}_k) \subseteq \mathcal{C}(P,\mathfrak{A})$$

  is satisfied for any net $(P,\mathfrak{A}_k)_{k\in K}$ of regular decision problems of $\tilde{\mathscr{D}}_\Theta$ and for any regular decision problem $(P,\mathfrak{A}) \in \tilde{\mathscr{D}}_\Theta$ where $\mathfrak{A}_k \searrow \mathfrak{A}$ holds.

[8] Let $(X_k)_{k\in K}$ be a net of sets directed by the index set K. It is called antitone whenever $X_l \subseteq X_k$ is satisfied for any $k \leq l$. We write $X_k \searrow X$, if $(X_k)_{k\in K}$ is an antitone net and $\cap_{k\in K} X_k = X$ holds.

The property of non-emptiness postulates that every regular, non-degenerated decision problem of $\tilde{\mathscr{D}}_\Theta$ is solvable, i.e., contains at least one favorable payoff profile. The property of closedness requires that the set of favorable payoff profiles of any regular decision problem of $\tilde{\mathscr{D}}_\Theta$ be a closed subset of $\mathbf{B}(\Theta)$. To understand the property of possibility set continuity from above, consider a payoff profile favorable in any decision problem of a family of regular decision problems having the same condition space and constraint, but whose possibility sets might be different. Then this property says that this payoff profile is also favorable in the decision problem having the same condition space and constraint, but whose possibility set corresponds to the intersection of the possibility sets of those decision problems. To understand the property of constraint continuity from above, consider a payoff profile favorable in any decision problem of a family of regular decision problems having the same condition space and possibility set, but whose constraints might be different. Then this property means that this payoff profile is also favorable in the decision problem having the same condition space and possibility set, but whose constraint corresponds to the intersection of the constraints of those decision problems. A choice rule is called *continuous from above in* $\tilde{\mathscr{D}}_\Theta$ whenever it is possibility set as well as constraint continuous from above in $\tilde{\mathscr{D}}_\Theta$ and is called *regular in* $\tilde{\mathscr{D}}_\Theta$ whenever it is non-empty, closed, and continuous from above. It turns out that with the exception of the choice rule of strict dominance all choice rules discussed above are regular in every system of decision problems. This result is summarized in the following remark.

Remark 2.3 *Consider some compact condition space Θ. The choice rules $\mathcal{MM}$, $\mathcal{PR}$, $\mathcal{SD}_+$, $\mathcal{SU}_m$, and $\mathcal{SU}_p$ are regular in $\mathscr{D}_\Theta$. Choice rule $\mathcal{SD}$ is closed and continuous in $\mathscr{D}_\Theta$ but not necessarily non-empty.*

Next, we aim to decomposing strategic games into decision problems. As discussed in the Introduction, the specification of the decision problems originating from a strategic game depends on the chosen approach to solving the game, i.e., whether the standard approach or the epistemic approach is selected. Nevertheless, with regard to certain aspects, these specifications are similar in both approaches.

Consider a player $i \in I$ participating in strategic game $\Gamma := (I, (S^i, z^i)_{i\in I})$. Let Θ be her, however specified, condition space and $P \subseteq \Theta$ her, however specified, possibility set. In order to determine her constraint, strategies must be converted into payoff profiles. We do this by means of two mappings. The first mapping is a so-called *strategy function* $\sigma_\Theta^{-i} : \Theta \to S^{-i}$, which assigns a strategy profile $s_\theta^{-i} \in S^{-i}$ of i's opponents to any condition $\theta \in \Theta$. This mapping might be understood as one that gives, for each condition θ, the strategies s_θ^{-i}, which i's opponents choose in this condition. The second mapping, which is termed the *payoff profile function*, is given by $\mathfrak{a}_\Theta^i : S^i \to \mathbf{B}(\Theta)$ where payoff profile $\mathfrak{a}_\Theta^i(s^i) := z^i(s^i; .) \circ \sigma_\Theta^{-i} \in \mathbf{B}(\Theta)$ is assigned to strategy $s^i \in S^i$. Payoff profile $\mathfrak{a}_\Theta^i(s^i)$ is called the *payoff profile on* Θ *induced by strategy* s^i. Obviously, if condition space Θ is compact and strategy mapping σ_Θ^{-i} as well as payoff function $z^i(s^i; .)$ are continuous, then payoff profile $\mathfrak{a}_\Theta^i(s^i)$ is continuous and closed.

Remark 2.4 *Consider a strategic game $\Gamma := (I, (S^i, z^i)_{i\in I})$. If condition space Θ is compact and strategy function σ_Θ^{-i} as well as payoff function z^i are continuous, then mapping $\mathfrak{a}_\Theta^i$ is continuous and closed.*

With above specifications at hand, it is possible to set up the decision problem of above player i. Her decision problem is representable by tuple $\left(P, \mathfrak{a}_\Theta^i(S^i)\right)_\Theta$. If for some reason the set of feasible strategies for player i is narrowed to subset $R^i \subseteq S^i$, then her decision problem is given by tuple $\big(P, \mathfrak{a}^i_\Theta(R^i)\big)_\Theta$.

A strategy s^i is called *favorable in decision problem* $\Phi^i_\Theta := \big(P, \mathfrak{a}^i_\Theta(R^i)\big)_\Theta$ whenever $\mathfrak{a}^i_\Theta(s^i) \in \mathcal{C}^i(P, \mathfrak{a}^i_\Theta(R^i))$ applies and *unfavorable in decision problem* Φ^i_Θ whenever $\mathfrak{a}^i_\Theta(s^i) \in \mathfrak{a}^i(R^i)$ but $\mathfrak{a}^i_\Theta(s^i) \notin \mathcal{C}^i(P, \mathfrak{a}^i_\Theta(R^i))$ applies. Denote by $(\mathfrak{a}^i_\Theta)^{-i}$ the inverse of mapping $\mathfrak{a}^i_\Theta$. Obviously, strategy s^i is favorable if and only if $s^i \in \big(\mathfrak{a}^i_\Theta\big)^{-i}\big(\mathcal{C}^i(P, \mathfrak{a}^i_\Theta(R^i))\big)$ holds. To advance the readability of our formal analysis, we slightly abuse our notation and write $\mathcal{C}^i(P, R^i)$ instead of $(\mathfrak{a}^i_\Theta)^{-i}\big(\mathcal{C}^i(P, \mathfrak{a}^i_\Theta(R^i))\big)$. Thus, whether the choice set is expressed as acts or as strategies is indicated by the constraint argument. If this describes a set of acts, then the first case holds. If it represents a set of strategies, the second case applies.

As mentioned in the Introduction, the above setup corresponds essentially to that of Apt and Zvesper (2010). Like them, we have taken choice rules as the starting point of our decision theoretic analysis of strategic games. Nevertheless, other approaches are possible. Unlike our approach, the decision theoretic analysis carried out by Epstein (1997) is based on preference relations. We conclude this section by arguing that Epstein's approach can be embedded into our setup without difficulties. To accomplish this task, it suffices to express Epstein's concept of preference rationalizability by some choice rule.

As before, let Θ be the condition space of a player $i \in I$, participating in strategic game $\Gamma := (I, (S^i, z^i)_{i\in I})$. The basic premise of Epstein (1997) is that the player is endowed with a *(weak) preference relation* $\succsim$ on the set $\mathbb{R}^\Theta$ of all payoff profiles on Θ. As usual, a preference relation $\succsim$ on $\mathbb{R}^\Theta$ represents a binary complete and transitive relation on $\mathbb{R}^\Theta$ and is interpreted as follows. Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^\Theta$ be some payoff profiles, then $\mathfrak{a} \succsim \mathfrak{b}$ is read as "the decision maker considers payoff profile $\mathfrak{a}$ at least as good as payoff profile $\mathfrak{b}$." A subset $E \subseteq \Theta$ of the condition space is called a *proposition*, and the complement of some proposition E is denoted by $\neg E$.[9] A proposition $E \subseteq \Theta$ is said to be *(Savage-)null* under preference relation $\succsim$ whenever $(\mathfrak{a}_E, \mathfrak{a}_{\neg E}) \succsim (\mathfrak{b}_E, \mathfrak{a}_{\neg E})$ holds for any payoff profiles $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}^\Theta$. That is, a proposition E is null whenever the decision maker is indifferent between any two payoff profiles yielding identical payoffs in conditions outside of proposition E.

In Epstein (1997), assumptions about the players' preference relations are summarized in so-called *preference models*. A preference model $\mathscr{P}$ is a correspondence assigning to every proposition $E \subseteq \Theta$ of any condition space Θ a set $\mathscr{P}(E)_\Theta$ of preference relations on $\mathbb{R}^\Theta$, where for each of them proposition $\neg E$ must be null. Armed with these models, the decision problems of the players are solved as follows. Consider the decision problem $\big(P, \mathfrak{a}^i_\Theta(S^i)\big)_\Theta$ of player i whose possibility set is given by $P \subseteq \Theta$ and some preference model $\mathscr{P}$. An available payoff profile $\mathfrak{a} \in \mathfrak{a}^i_\Theta(S^i)$ is said to be *rationalizable by preference model* $\mathscr{P}$ whenever there exists a preference relation $\succsim \in \mathscr{P}(P)_\Theta$ so that $\mathfrak{a} \succsim \mathfrak{b}$ holds for every payoff profile $\mathfrak{b} \in \mathfrak{a}^i_\Theta(S^i)$. The main issue addressed in Epstein (1997) is the detection of a general preference model so that the solution obtained by the iterated application of the concept of preference rationalizability on some strategic game coincides with the solution if

[9] We remark that this terminology is not standard. Generally, in decision theory subsets of the condition space are called events. We deviate from the standard terminology since we use the term 'event' for subsets of state spaces (see Section 4). While a condition space describes only the uncertainty of one player, a state space summarizes the uncertainty of all players.

Condition space Θ

Payoff profiles		θ_1	θ_2
	$\mathfrak{a}$	1	1
	$\mathfrak{b}$	1	0
	$\mathfrak{c}$	0	0

Figure 2: Condition space $\Theta := \{\theta_1, \theta_2\}$ with payoff profiles $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$

the players of this game act rationally according to these preference models and this is commonly believed among them.

To bring Epstein's concept of preference rationalizability in line with our setup, we introduce the choice rule $\mathcal{C}_{\mathscr{P}}$ specified by

$$\mathcal{C}_{\mathscr{P}}(P, \mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A} : \text{ there exists } \succsim \in \mathscr{P}(P)_\theta \text{ so that } \mathfrak{a} \succsim \mathfrak{b} \text{ for every } \mathfrak{b} \in \mathfrak{A}\}$$

for every decision problem $(P, \mathfrak{A})_\Theta$ of any condition space Θ. This is called the *choice rule induced by preference model* $\mathscr{P}$. Obviously for every decision problem it gives the set of payoff profiles rationalizable by possibility model $\mathscr{P}$. Thus, we have transformed the concept of preference rationalizability into a choice rule as desired.

We note that not any choice rule can be induced by some preference model. A trivial example supporting this claim is the modified strict dominance rule $\mathcal{SD}_+$ introduced above. To see this, consider decision problems whose common condition space is given by $\Theta := \{\theta_1, \theta_2\}$. To be compact Hausdorff, this space must be endowed with the discrete topology. In Figure 2, payoff profiles $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ on Θ are specified.

Obviously, we obtain two results, $\mathcal{SD}_+(\Theta, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}) = \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ and $\mathcal{SD}_+(\Theta, \{\mathfrak{a}, \mathfrak{c}\}) = \{\mathfrak{a}\}$. According to the latter result, every preference relation $\succsim \in \mathscr{P}(\Theta)_\Theta$ of any preference model $\mathscr{P}$ rationalizing choice rule $\mathcal{SD}_+$ must satisfy $\mathfrak{a} \succ \mathfrak{c}$. Consequently, $\mathfrak{c} \notin \mathcal{SD}_+(\Theta, \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}\})$ should hold. However, this contradicts our former result. Remarkably, although the modified strict undominance rule is not rationalizable by any preference model, we are still able to provide a plausible choice-rule based epistemic rationale for its iterated application on the class of regular strategic games (see our Remark 5.7).

3 Iterated Application of Choice Rules

In this section, we study solution concepts originating from iterated applications of choice rules. Two issues regarding these solution concepts are addressed. First, we tackle the problem of existence and search for properties of choice rules ensuring that these solution concepts provide a solution at least for each regular strategic game. This problem is solved finally in Remark 3.2. Second, we address the problem of stability. Stability is understood here in two ways. Our first version of stability requires that the solution obtained by iterated applications of choice rules be irreducible with respect to further applications of choice rules. This kind of stability is referred to as *irreducibility*. Our second version requires that the strategies of a player surviving the iterated application of choice rules still be favorable in the decision problem where every strategy of the player is feasible, but only those strategies of the opponents are considered possible that survive the iterated application of choice rules. If a solution concept exhibits such stability, it is said to have the *best choice property*. The best choice property is a generalization of the best response property, which has been introduced by Pearce (1984) with regard to his solution concept of rationalizability. Our task is to detect properties of choice rules so that the solution generated by iterated applications of choice rules is stable in the above two senses. The answers to these issues are summarized in Theorem 3.6 and 3.10, respectively.

As explained in the previous section, implementing choice rules on a strategic game requires to decompose it into decision problems. According to our framework set out there, a decision problem is described by the three attributes, namely, condition space, possibility space, and constraint. In this section, these attributes are specified as follows. Let $\Gamma|_R$ be a reduction of strategic game $\Gamma := (I, (S^i, z^i)_{i \in I})$, then the decision problem for player i is determined by

$$\Phi^i_{\Gamma|_R} := (R^{-i}, \mathfrak{a}^i_\Gamma(R^i))_{S^{-i}} \in \mathscr{D}_{S^{-i}} \ ,$$

where mapping $\mathfrak{a}^i_\Gamma$ assigns to each strategy s^i of player i the payoff profile $z^i(s^i;.)$ on S^{-i}. By reviewing this specification, we recognize that its condition space contains all conceivable profiles of strategies of i's opponents, its possibility set contains all profiles of strategies of i's opponents available in the reduced game $\Gamma|_R$, and its constraint contains all payoff profiles induced by i's strategies available in the reduced game $\Gamma|_R$. Henceforth, decision problem $\Phi^i_{\Gamma|_R}$ is referred to as the *decision problem for player i in the reduced game* $\Gamma|_R$. The set of all decision problems for player i, which are induced by some reduced game of Γ, is denoted by $\mathscr{D}^i_\Gamma$.

If choice rule $\mathcal{C}^i$ is applied to solving decision problem $\Phi^i_{\Gamma|_R}$, then $\left(\mathfrak{a}^i_{\Gamma|_R}\right)^{-1}\left(\mathcal{C}^i(\Phi^i_{\Gamma|_R})\right)$ represents the set of strategies of player i that are considered favorable according to choice rule $\mathcal{C}^i$. Resorting to the rule of simplification suggested in the previous section, the latter term can be rewritten as $\mathcal{C}^i(R^{-i}, R^i_k)_{S^{-i}}$. With this simplified notation at our hands, the solution concept of iterated deletion of choice rules is defined as follows.

Definition 3.1 *Consider a strategic game Γ and let $\mathcal{C} := (\mathcal{C}^i)_{i \in I}$ be a family of choice rules. The deletion process of iterated application of choice rules $\mathcal{C}$ on Γ is a sequence of $(R_k)_{k \in \mathbb{N}_0}$ of sets of strategy profiles, which is recursively determined by $R_0 := S$ and*

$$R_{k+1} := \prod_{i \in I} \mathcal{C}^i(R^{-i}_k, R^i_k)_{S^{-i}}$$

for all $k \geq 0$. We say about strategy profile $s \in R_k$ that it survives k rounds of deletion of $\mathcal{C}$-unfavorable strategy profiles. Intersection

$$R_\infty := \bigcap_{k \in \mathbb{N}_0} R_k$$

is referred to as the set of strategy profiles surviving the iterated application of choice rules $\mathcal{C}$ on Γ.

Reviewing Definition 3.1, we recognize that the deletion process of iterated application of choice rules relies only on the objective features of a game (i.e., the features listed in Definition

2.1). Accordingly, this construction of a solution concept is in line with what we called earlier the standard approach for solving games.

Furthermore, we notice that, according to Definition 3.1, in each round of this process every strategy considered unfavorable is deleted. Such deletion processes are known as *processes of maximal deletion of unfavorable strategies*. From now on, if family $\mathcal{C} := (\mathcal{C}^i)_{i \in I}$ of choice rules is fixed, then the solution concept that assigns to each strategic game the set of strategy profiles surviving the iterated application of $\mathcal{C}$ is denoted by $\mathrm{IA}_{\mathcal{C}}$. The following remark states properties of choice rules so that $\mathrm{IA}_{\mathcal{C}}$ provides a non-empty solution at least for every regular strategic game.

Remark 3.2 *Consider a regular strategic game Γ and let $(R_k)_{k \in \mathbb{N}_0}$ be the deletion process generated by iterated application of choice rules $\mathcal{C} := (\mathcal{C}^i)_{i \in I}$. If each choice rule $\mathcal{C}^i$ is non-empty and closed in $\mathscr{D}^i_\Gamma$, then*

(a) R_k is a non-empty and closed restriction of S for every $k \in \mathbb{N}_0$.

(b) $R_\infty := \bigcap_{k \in \mathbb{N}_0} R_k$ is a non-empty and closed restriction of S.

In order to tackle the two problems of stability we have mentioned at the outset of this section, further properties of choice rules have to be imposed.

Definition 3.3 *Let Θ be some condition space and $\tilde{\mathscr{D}}_\Theta \subseteq \mathscr{D}_\Theta$ be some system of decision problems. A choice rule $\mathcal{C}$ is*

- *called reflexive in $\tilde{\mathscr{D}}_\Theta$ if*

$$\mathcal{C}(Q, \mathcal{C}(P, \mathfrak{A})) \subseteq \mathcal{C}(Q, \mathfrak{A})$$

holds for any regular decision problems $(P, \mathfrak{A}), (Q, \mathfrak{A}), (Q, \mathcal{C}(P, \mathfrak{A})) \in \tilde{\mathscr{D}}_\Theta$ where $Q \subseteq P$ applies.

- *satisfies Aizerman's property in $\tilde{\mathscr{D}}_\Theta$ if*

$$\mathcal{C}(P, \mathfrak{A}) \subseteq \mathcal{C}(P, \tilde{\mathfrak{A}})$$

holds for any regular decision problems $(P, \mathfrak{A}), (P, \tilde{\mathfrak{A}}) \in \tilde{\mathscr{D}}_\Theta$ where $\mathcal{C}(\Phi) \subseteq \tilde{\mathfrak{A}} \subseteq \mathfrak{A}$ applies.

- *called monotone in $\tilde{\mathscr{D}}_\Theta$ if*

$$\mathcal{C}(Q, \mathfrak{A}) \subseteq \mathcal{C}(P, \mathfrak{A})$$

holds for any regular decision problems $(P, \mathfrak{A}), (Q, \mathfrak{A}) \in \tilde{\mathscr{D}}_\Theta$ where $Q \subseteq P$ applies.

To understand the property of reflexivity, consider two regular decision problems with the same condition space and constraint, but where one has a smaller possibility set. Suppose there is a payoff profile that is unfavorable in the decision problem with the smaller possibility set. Then the property of reflexivity signifies that this payoff profile must be either unavailable or unfavorable in the decision problem with the same condition space and possibility set, but with a constraint consisting of all payoff profiles that are favorable in the decision problem with the larger possibility set. Aizerman's property requires that favorable payoff profiles remain favorable even if unfavorable payoff profiles are omitted from constraint. The property of monotonicity demands that a payoff profile viewed as favorable in some decision problem is also favorable in every decision problem with the same condition space and constraint but a larger possibility set. Without difficulty, it can be shown that with the exception of the maximin rule all choice rules discussed above satisfy the above three properties for any complete system $\mathscr{D}_\Theta$ of decision problems. For the maximin rule, only Aizerman's property is guaranteed for any complete system $\mathscr{D}_\Theta$ of decision problems. As the two examples of decision problems in Figure 3 demonstrate, the maximin rule satisfies neither reflexivity nor monotonicity in general.

Decision problem under ...

condition space Θ_1 and payoff profiles

	ω_1	ω_2
$\mathfrak{t}$	0	3
$\mathfrak{m}$	2	1
$\mathfrak{d}$	1	2

$\mathfrak{A} := \{\mathfrak{t}, \mathfrak{m}, \mathfrak{d}\},\ P := \{\omega_1, \omega_2\},\ Q := \{\omega_2\},$
$\mathcal{MM}(Q, \mathcal{MM}(P, \mathfrak{A})) = \{\mathfrak{d}\} \nsubseteq \{\mathfrak{t}\} = \mathcal{MM}(Q, \mathfrak{A})$

condition space Θ_2 and payoff profiles

	ω_1	ω_2	ω_3
$\mathfrak{l}$	1	2	0
$\mathfrak{r}$	0	1	2

$\mathfrak{A} := \{\mathfrak{l}, \mathfrak{r}\},\ P := \{\omega_2, \omega_3\},\ Q := \{\omega_2\},$
$\mathcal{MM}(Q, \mathfrak{A}) = \{\mathfrak{l}\} \nsubseteq \{\mathfrak{r}\} = \mathcal{MM}(P, \mathfrak{A})$

Figure 3: Maximin rule applied to decision problems under condition spaces Θ_1 and Θ_2

In the following remark, the properties of the choice rules considered in Section 2 are summarized.

Remark 3.4 *Consider some condition space Θ. Choice rules $\mathcal{PR}$, $\mathcal{SD}_+$, $\mathcal{SU}_m$ and $\mathcal{SU}_p$ satisfy reflexivity, Aizerman's property and monotonicity in $\mathscr{D}_\Theta$. Choice rule $\mathcal{MM}$ satisfies Aizerman's property in $\mathscr{D}_\Theta$ but not necessarily reflexivity and monotonicity.*

In the succeeding paragraphs, we deal with the question under which conditions the solution obtained by iterated application of choice rules is irreducible (or, synonymously, stable) with respect to further applications of these rules. To resolve this issue, the following lemma turns out to be helpful.

Lemma 3.5 *Consider a regular strategic game Γ. Let $(R_k)_{k\in\mathbb{N}_0}$ be the deletion process generated by iterated application of choice rules $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ and R_∞ be its solution. If each choice rule $\mathcal{C}^i$ is non-empty, closed, possibility set continuous from above, and reflexive in $\mathscr{D}^i_\Gamma$, then $R^i_\infty \subseteq \mathcal{C}^i(R^{-i}_\infty, R^i_k)$ holds for any $k \in \mathbb{N}_0$ and for any $i \in I$.*

Proof. Pick some player $i \in I$ and fix some $l \geq 0$. We show by induction on k that $\mathcal{C}^i(R^{-i}_l, R^i_l) \subseteq \mathcal{C}^i(R^{-i}_l, R^i_k)$ holds for every $0 \leq k \leq l$ and every $i \in I$. Case $k = l$ is trivial. Suppose that $\mathcal{C}^i(R^{-i}_l, R^i_l) \subseteq \mathcal{C}^i(R^{-i}_l, R^i_k)$ is satisfied for some $0 < k \leq l$. It follows from the definition of R^i_k and the reflexivity of $\mathcal{C}^i$ that

$$\mathcal{C}^i(R^{-i}_l, R^i_k) = \mathcal{C}^i(R^{-i}_l, \mathcal{C}^i(R^{-i}_{k-1}, R^i_{k-1})) \subseteq \mathcal{C}^i(R^{-i}_l, R^i_{k-1})$$

holds. This result together with the induction premise implies $\mathcal{C}^i(R^{-i}_l, R^i_l) \subseteq \mathcal{C}^i(R^{-i}_l, R^i_{k-1})$, and the above claim has been proven. By construction, $R^i_\infty \subseteq \mathcal{C}^i(R^{-i}_l, R^i_l)$ applies to any $l \in \mathbb{N}_0$. Due to the result just proven, we obtain $R^i_\infty \subseteq \mathcal{C}^i(R^{-i}_l, R^i_k)$ for any $l, k \in \mathbb{N}_0$ satisfying $k \leq l$.

Hence, $R^i_\infty \subseteq \cap_{l=k}^{\infty} \mathcal{C}^i(R_l^{-i}, R_k^i)$ holds for any $k \in \mathbb{N}_0$. By possibility set continuity from above, $R^i_\infty \subseteq \mathcal{C}^i(\cap_{l=k}^{\infty} R_l^{-i}, R_k^i) = \mathcal{C}^i(R_\infty^{-i}, R_k^i)$ is obtained for any $k \in \mathbb{N}_0$ as desired. □

Consider a strategic game Γ and a family $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ of choice rules. *Dominance operator* $\mathcal{D}^{\mathcal{C}}_\Gamma : \mathscr{R}_\Gamma \to \mathscr{R}_\Gamma$ is a mapping that associates with each restriction $R \in \mathscr{R}_\Gamma$ restriction

$$\mathcal{D}^{\mathcal{C}}_\Gamma(R) := \prod_{i\in I} \mathcal{C}^i(R^{-i}, R^i) \; .$$

Obviously, $\mathcal{D}^{\mathcal{C}}_\Gamma(R)$ is the set of all strategy profiles whose components are favorable strategies under restriction R and choice rules $\mathcal{C}$. A *fixed point* of dominance operator $\mathcal{D}^{\mathcal{C}}_\Gamma$ is a restriction $R \in \mathscr{R}_\Gamma$ having property $\mathcal{D}^{\mathcal{C}}_\Gamma(R) = R$. If a restriction proves to be a fixed point of mapping $\mathcal{D}^{\mathcal{C}}_\Gamma(R)$, this means nothing but that this solution is not further reducible with respect to choice rules $\mathcal{C}$. The following theorem states properties of choice rules ensuring that solution $\mathrm{IA}_{\mathcal{C}}(\Gamma)$ is a non-empty fixed point of dominance operator $\mathcal{D}^{\mathcal{C}}_\Gamma$.

Theorem 3.6 *Consider a regular strategic game Γ and let R_∞ be the solution generated by iterated applications of choice rules $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$. If each choice rule $\mathcal{C}^i$ is regular and reflexive in $\mathscr{D}^i_\Gamma$, then R_∞ is a non-empty fixed point of dominance operator $\mathcal{D}^{\mathcal{C}}_\Gamma$.*

Proof. It follows immediately from the definition of the dominance operator $\mathcal{D}^{\mathcal{C}}_\Gamma$ that $\mathcal{D}^{\mathcal{C}}_\Gamma(R_\infty) \subseteq R_\infty$. It remains to show $\mathcal{D}^{\mathcal{C}}_\Gamma(R_\infty) \supseteq R_\infty$. This statement is proved by the following line of argumentation.

$$\begin{aligned}
\mathcal{D}^{\mathcal{C}}_\Gamma(R_\infty) &= \prod_{i\in I} \mathcal{C}^i(R_\infty^{-i}, R_\infty^i) && \text{(by definition of } \mathcal{D}^{\mathcal{C}}_\Gamma\text{)}\\
&= \prod_{i\in I} \mathcal{C}^i(R_\infty^{-i}, \bigcap_{k\in\mathbb{N}_0} R_k^i) && \text{(by definition of } R^i_\infty\text{)}\\
&\supseteq \prod_{i\in I} \bigcap_{k\in\mathbb{N}_0} \mathcal{C}^i(R_\infty^{-i}, R_k^i) && \text{(by constraint continuity from above)}\\
&\supseteq \prod_{i\in I} R^i_\infty && \text{(by Lemma 3.5)}\\
&= R_\infty && \text{(by definition of } R_\infty\text{)}
\end{aligned}$$

□

The remaining part of this section is devoted to the second variant of stability we discussed at the outset of this section. This kind of stability is termed the best choice property and is a generalization of the best response property of Pearce (1984).

Definition 3.7 *Consider a strategic game Γ and family $(\mathcal{C}^i)_{i\in I}$ of choice rules . A restriction $R \in \mathscr{R}_\Gamma$ has the best choice property if $R^i \subseteq \mathcal{C}^i(R^{-i}, S^i)$ is satisfied for any player $i \in I$.*

In words, a restriction of a game satisfies the best choice property whenever, for each player, each strategy of this restriction is favorable in the decision problem composed of the following possibility set and constraint. The possibility set consists exactly of the opponents' strategies being available in the restriction, and the constraint contains all the player's strategies being available in this game. In the case that the players' choice rules are those that select the strategies maximizing the expected payoff for some product measure on the opponents' restriction (i.e., the product of measures where each of them is defined on the restriction of one opponent), the best choice property turns into the best response property of Pearce (1984). In the following, we state properties of choice rules so that the solution generated by iterated application of choice rules has the best choice property. To accomplish this, we resort to the following lemma.

Lemma 3.8 *Consider a regular strategic game* $\Gamma := (I, (S^i, z^i)_{i \in I})$ *and let* $(R_k)_{k \in \mathbb{N}_0}$ *be the deletion process generated by iterated application of choice rules* $\mathcal{C} := (\mathcal{C}^i)_{i \in I}$. *If each choice rule* $\mathcal{C}^i$ *is non-empty, closed, and satisfies Aizerman's property in* $\mathscr{D}^i_\Gamma$, *then* $\mathcal{C}^i(T^{-i}, S^i) \subseteq \mathcal{C}^i(T^{-i}, R^i_k)$ *is satisfied for every* $T^{-i} \subseteq R^{-i}_k$ *closed in* S^{-i}, *every* $k \in \mathbb{N}_0$, *and every* $i \in I$.

Proof. This claim is proved by induction on the deletion rounds. In the case of $k = 0$, this assertion follows directly from our stipulation $R^i_0 := S^i$. Suppose this assertion holds for some $k \in \mathbb{N}_0$. Consider some closed restriction $T^{-i} \subseteq R^{-i}_{k+1}$ and some player $i \in I$. We obtain

$$\begin{aligned}
\mathcal{C}^i(T^{-i}, R^i_{k+1}) &= \mathcal{C}^i(T^{-i}, \mathcal{C}^i(R^{-i}_k, R^i_k)) && \text{(by definition of } R^i_{k+1}) \\
&\supseteq \mathcal{C}^i(T^{-i}, R^i_k) && \text{(by Aizerman's property)} \\
&\supseteq \mathcal{C}^i(T^{-i}, S^i)\ , && \text{(by induction premise)}
\end{aligned}$$

and thus the above assertion is also established for $k + 1$. □

Lemma 3.9 *Consider a regular strategic game* Γ *and a family* $\mathcal{C} := (\mathcal{C}^i)_{i \in N}$ *of choice rules. If each choice rule* $\mathcal{C}^i$ *of* $\mathcal{C}$ *is non-empty, closed, and monotone, and satisfies Aizerman's property in* $\mathscr{D}^i_\Gamma$, *then every restriction of* Γ *having the best choice property survives the iterated applications of choice rules* $\mathcal{C}$.

Proof. Let $(R_k)_{k \in \mathbb{N}_0}$ be the deletion process generated by iterated applications of choice rules $\mathcal{C}$, and consider a restriction $R \in \mathscr{R}_\Gamma$ having the best choice property. We show by induction that $R \subseteq R_k$ applies to any $k \in \mathbb{N}_0$. Obviously, $R^i \subseteq R^i_0 := S^i$ is satisfied for any $i \in I$. Suppose $R^i \subseteq R^i_k$ holds for some $k \in \mathbb{N}_0$. We obtain

$$\begin{aligned}
R^i &\subseteq \mathcal{C}^i(R^{-i}, S^i) && \text{(by assumption } R \text{ has the best choice property)} \\
&\subseteq \mathcal{C}^i(R^{-i}_k, S^i) && \text{(by induction premise and monotonicity)} \\
&\subseteq \mathcal{C}^i(R^{-i}_k, R^i_k) && \text{(by Lemma 3.8)} \\
&= R^i_{k+1} && \text{(by definition of } R^i_{k+1})
\end{aligned}$$

for every player $i \in I$. Since $R \subseteq R_k$ holds for every $k \in \mathbb{N}_0$, our claim $R \subseteq \cap_{k \in \mathbb{N}_0} R_k = R_\infty$ is verified. □

By merging Lemmata 3.5 and 3.9, we are able to specify a set of choice rule properties guaranteeing that the solution generated by iterated application of choice rules is the largest restriction having best response property. This result is summarized in the following theorem.

Theorem 3.10 *Consider a regular strategic game* Γ *and let* R_∞ *be the solution generated by iterated applications of choice rules* $\mathcal{C} := (\mathcal{C}^i)_{i \in N}$. *If each choice rule* $\mathcal{C}^i$ *is non-empty, closed, possibility set continuous from above, reflexive and monotone, and satisfies Aizerman's property in* $\mathscr{D}^i_\Gamma$, *then* R_∞ *is the largest restriction satisfying the best choice property.*

Proof. Lemma 3.5 implies that $R^i_\infty \subseteq \mathcal{C}^i(R^{-i}_\infty, S^i)$ holds for every $i \in I$. That is to say, R_∞ has the best choice property. In order to establish that it is the largest restriction having this property, consider an arbitrary restriction R with the best choice property. By Lemma 3.9, restriction R survives the iterated applications of choice rules $\mathcal{C}$. Hence, $R \subseteq R_\infty$ results. □

As established in Remarks 2.3 and 3.4, except for the minimax rule all choice rules analyzed in Section 2 satisfy the assumptions of the theorems proved in the current section. Hence, the solution obtained by the iterated application of these choice rules exhibits the following set of properties.

Remark 3.11 *Consider a regular strategic game* Γ *and a family* $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ *of choice rules where* $\mathcal{C}^i \in \{\mathcal{PR}, \mathcal{SD}_+, \mathcal{SU}_p, \mathcal{SU}_m\}$ *holds for every* $i \in I$*. Then solution* $IA_\mathcal{C}(\Gamma)$ *is*

(a) non-empty and closed,

(b) irreducible with respect to further application of choice rules $\mathcal{C}$*,*

(c) is the largest restriction satisfying the best choice property.

We note that our last remark comprises results already known in the game theory community. For example, Bernheim (1984, Theorem 3.1) has already established that the solution obtained by the iterated application of the choice rule of point rationality satisfies the above properties *(a)* - *(c)* whenever the players' strategy sets are compact subsets of some Euclidean space $\mathbb{R}^n$ and their payoff functions are continuous.[10] Moreover, it follows immediately from Bernheim (1984, Theorem 3.2) and Pearce (1984, Lemma 3) that, for such strategic games and with two players, the solution obtained by iterated application of the choice rule of strict undominance in mixtures has the same properties.[11] The iterated application of the choice rule of strict undominance has been broadly analyzed in Dufwenberg and Stegeman (2002). Although their main goal is to establish the order independence of this iterative deletion procedure in general strategic games, they prove in their Theorem 1(b) that its solution satisfies the above properties, *(a)* and *(b)*, in regular strategic games.

[10] A generalization of this result that rests on assumptions even milder than the ones we postulate can be found in Ok (2004). He shows in his Theorem 6.1 that properties *(a)* - *(c)* are already satisfied for strategic games in which the strategy set S^i of each player $i \in I$ is a compact Hausdorff space and i's payoff function z^i is upper semicontinuous so that i's value function $z^*_i(s_{-i}) := \max_{s_i \in S_i} z_i(s_i, s_{-i})$ is lower semicontinuous.

[11] The reason why their findings do not ensure these properties for any arbitrary finite number of players is their assumption that each player has uncorrelated probabilistic beliefs about their opponents' choices. That is, the probability measure representing the player's belief about the opponents' choices is a product of probability measures, where each of them represents the player's belief about the choices of one opponent. However, a strategy turns out to be favorable under the choice rule of strict undominance in mixtures if and only if this strategy maximizes the players' expected payoff for some (possibly correlated) probabilistic belief about the opponents' choices. Therefore, in order to deduce that the solution resulting from the iterated application of this choice rule satisfies the above properties, *(a)* - *(c)*, in strategic games with more than two players, arbitrary probabilistic beliefs must be considered. Fortunately, by slight modifications of the arguments put forward in Bernheim (1984) and Pearce (1984) such generalization can be established.

4 Common Belief of Applying Choice Rules

In the previous section, we have taken up the standard approach for solving strategic games. The solution algorithms of iterated application of choice rules put forward there are based only on the objective features of the game (i.e., those listed in Definition 2.1). In this section, we pursue a different - an epistemic - approach for solving games. Instead of specifying a solution concept ad hoc as above, this approach requires to view the game from the perspective of the players and to reason their choices from their conjectures about the opponents' choices and conjectures. Since there is usually no objective device that assigns likelihoods to the latter choices, these conjectures might be of subjective nature. Unlike the traditional ad hoc approach on games, the epistemic approach takes into account this subjective feature of a game.

To put ourselves in the position of the players and to describe their decision problems, we adopt an idea of Harsanyi (1967/68) and supplement strategic games with type space models. As will be seen, this supplementation enables us to decompose strategic games into individual decision problems under subjective uncertainty and to scrutinize the decision makings of the players.

Definition 4.1 *A type space model to strategic game* $\Gamma := (I, (S^i, z^i)_{i \in I})$ *is a tuple* $\mathbb{T} := (T^i, P^i)_{i \in I}$ *where*

- *topological space* T^i *denotes the type space of player* i*.*
- *correspondence* $P^i : T^i \twoheadrightarrow \prod_{j \in I \setminus \{i\}} (S^j \times T^j)$ *denotes the possibility correspondence of player* i *where* $P^i(t^i)$ *is non-empty for every* $t_i \in T_i$*.*

A type space model is called regular if, for each player $i \in I$*, type space* T^i *is compact Hausdorff and possibility correspondence* P^i *is continuous (i.e., upper as well as lower hemicontinuous) and closed-valued.*

Different to the original setup of Harsanyi (1967/68), our type space model is only of qualitative nature. In Harsanyi's type space model, each type of a player is associated with a probability measure on the opponents' strategy-type combinations. Such measures quantify the types' degrees of beliefs on the opponents' strategy choices and types. Noteworthy, such quantification of the types' beliefs is not postulated in our above definition of a type space model. Rather, we pursue a more general approach and just associate each type of a player with a possibility set indicating the set of opponents' strategy-type combinations considered possible by this type. In the epistemic game theory literature, such qualitative type space models are known as *possibility structures*. Mariotti et al. (2005) analyze such structures in depth. Among others, they establish the existence of a universal regular type space model $\mathbb{T}^*$ for any regular strategic game Γ. That is, each regular type space model $\mathbb{T}$ to the strategic game Γ can be uniquely embedded into the type space model $\mathbb{T}^*$ in the sense that the beliefs of each type of type space model $\mathbb{T}$ is preserved by this embedding.

Henceforth, pair $\mathbb{T}_\Gamma := (\Gamma, \mathbb{T})$, consisting of strategic game Γ and type space model $\mathbb{T}$ of it, is referred to as a strategic game framed by a type space or simply a *framed strategic game*. If the framed strategic game $\mathbb{T}_\Gamma := (\Gamma, \mathbb{T})$ consists of a regular strategic game and a regular type space model, we call it a *regular framed strategic game*. As usual, we write $T := \prod_{i \in I} T^i$ for the set of profiles specifying the types of all players and $T^{-i} := \prod_{j \in I \setminus \{i\}} T^j$ for the set of profiles specifying only the types of player i's opponents.

Consider some framed strategic game $\mathbb{T}_\Gamma$. The product set $\Omega := \prod_{i \in I}(S^i \times T^i)$, which is endowed with the product topology, constitutes the *state space* of it. In the case that $\mathbb{T}_\Gamma$ is regular, the topology of Ω is Hausdorff and, by Tychonoff Product Theorem, also compact. An element of state space Ω is called *state of the world* and represents a specific resolution of the players' strategy-type combinations. Hereafter, states of the world are represented by Greek lower case letters (usually, by ω) and state spaces by Greek capital letters (usually, by Ω).

The players' strategy spaces S^i and type spaces T^i are called factors of state space Ω. A *subspace of* $\Omega := \prod_{i \in I}(S^i \times T^i)$ is a product set in which some or none of the factors of Ω are omitted. Throughout this paper, we assume that any subspaces of Ω are endowed with the product topology. Obviously, if a regular framed strategic game is considered, these subspaces turn out to be compact Hausdorff. Let I_S and I_T be subsets of I where at least one of them is non-empty. Then $\Omega^{(I_S, I_T)}$ denotes the subspaces of Ω given by

$$\Omega^{(I_S, I_T)} := \prod_{\substack{i \in I_R \\ R \in \{S, T\}}} R^i \; .$$

Furthermore, suppose $I_S \subseteq J_S \subseteq I$ and $I_T \subseteq J_T \subseteq I$ hold. We denote by $\text{proj}_{\Omega^{(I_S,I_T)}}^{\Omega^{(J_S,J_T)}} : \Omega^{(J_S,J_T)} \to \Omega^{(I_S,I_T)}$ the projection of subspace $\Omega^{(J_S,J_T)}$ into subspace $\Omega^{(I_S,I_T)}$. The following remark states that these projections are closed if the framed strategic game is regular.

Remark 4.2 *Consider a framed strategic game* $\mathbb{T}_\Gamma$ *and sets* $I_S \subseteq J_S \subseteq I$ *and* $I_T \subseteq J_T \subseteq I$ *where* I_S *or* I_T *is non-empty.*

(a) Then projection $\text{proj}_{\Omega^{(I_S,I_T)}}^{\Omega^{(J_S,J_T)}}$ *is surjective and continuous.*

(b) If $\mathbb{T}_\Gamma$ *is regular, then* $\text{proj}_{\Omega^{(I_S,I_T)}}^{\Omega^{(J_S,J_T)}}$ *is closed.*

To simplify our notation, we omit superscript $\Omega^{(J_S,J_T)}$ of a projection $\text{proj}_{\Omega^{(I_S,I_T)}}^{\Omega^{(J_S,J_T)}}$ whenever the domain of this projection corresponds to state space Ω. Some subspaces of Ω are specifically marked. The subspace consisting of the strategy-type combinations of player i is denoted by $\Omega^i := S^i \times T^i$ and the subspace consisting of the profiles of strategy-type combinations of i's opponents by $\Omega^{-i} := \prod_{j \in I \setminus \{i\}}(S^j \times T^j)$. Let $\omega \in \Omega$, then $\omega^i := \text{proj}_{\Omega^i}(\omega)$ gives the strategy-type combination of player i at state ω, and $\omega^{-i} := \text{proj}_{\Omega^{-i}}(\omega)$ gives the strategy-type combinations of i's opponents at state ω. Moreover, we denote by $t_\omega := \text{proj}_T(\omega)$ the type profile at state ω, by $s_\omega := \text{proj}_S(\omega)$ the strategy profile chosen at state ω, by $t^i_\omega := \text{proj}_{T^i}(\omega)$ the type of player i at state ω, and by $s^i_\omega := \text{proj}_{S^i}(\omega)$ the strategy chosen by player i at state ω.

As is standard in the theory of decision under uncertainty, any subset of state space Ω is termed *event*. An event $E \subseteq \Omega$ is called closed (compact) if it is a closed (resp. compact) subset of Ω, and rectangular if $E = \prod_{i \in I} E^i$ holds where each E^i is a subset of subspace Ω^i. Consider some state $\omega \in \Omega$ and some player $i \in I$. A profile $\tilde{\omega}^{-i} \in \Omega^{-i}$ of strategy-type combinations of i's opponents is said to be *compatible in event* E *with* i*'s strategy-type combination* ω^i whenever $(\omega^i, \tilde{\omega}^{-i}) \in E$ holds. Henceforth, the set of those profiles is denoted by $Q^i_E(\omega^i) := \{\tilde{\omega}^{-i} \in \Omega^{-i} : (\omega^i, \tilde{\omega}^{-i}) \in E\}$. This set is also known as the ω^i cross section of event E. Hereinafter, $Q^i_E : \Omega^i \to 2^{\Omega^{-i}}$ denotes the correspondence assigning to each strategy-type combination $\omega^i \in \Omega^i$ of player i its cross section $Q^i_E(\omega^i)$ of event E.

Remark 4.3 *Consider a regular framed strategic game $\mathbb{T}_\Gamma$, a closed event $E \subseteq \Omega$, and a player $i \in I$. Then correspondence Q^i_E is closed-valued and upper hemicontinuous.*

According to Definition 4.1, a framed strategic game assigns to each player $i \in I$ a possibility correspondence P^i that for each state $\omega \in \Omega$ determines a subset $P^i(t^i_\omega)$ of Ω^{-i}. The latter set is termed the *possibility set of player i at state ω*. It is interpreted as the set of profiles of strategy-type combinations of i's opponents deemed possible by i. From this possibility set, player i's beliefs about events are deduced. A player i is said to *believe event E at state ω* whenever each profile of the opponents' strategy-type combinations deemed possible by her at state ω is compatible in E with her actual strategy-type combination ω^i. Or in formal terms, player i believes event E at state ω whenever $P^i(t^i_\omega) \subseteq Q^i_E(\omega^i)$ is satisfied. From player i's possibility correspondence, a *belief operator of player i* is constructed. It is the mapping $B^i : 2^\Omega \to 2^\Omega$ satisfying

$$B^i(E) := \{\omega \in \Omega : P^i(t^i_\omega) \subseteq Q^i_E(\omega^i)\}$$

for every event $E \subseteq \Omega$. Due to this definition, player i believes event E at state ω if and only if $\omega \in B^i(E)$ holds. The following remark states that in regular framed strategic games the set of states in which the closed event E is believed is also a closed event.

Remark 4.4 *Consider a regular framed strategic game $\mathbb{T}_\Gamma$ and a player $i \in I$. If event E is closed, then event $B^i(E)$ is closed.*

Let $E \subseteq \Omega$ be an event of framed strategic game $\mathbb{T}_\Gamma$. We recursively define a sequence $(E_k)_{k\in\mathbb{N}_0}$ of events where $E_0 := E$ and

$$E_{k+1} := E_k \cap \left(\bigcap_{i\in I} B^i(E_k)\right)$$

holds for any $k \in \mathbb{N}_0$. Obviously, event E_1 is the event in which event E holds and this is believed by every player. In other words, E_1 represents the event in which E is true and there is *first order (or mutual) belief of E*. If there is first order belief of E and every player believes that there is first order belief of E, then this is referred to as the *second order belief of E*. Obviously, event E_2 represents the event in which E is true and there is second order belief of E. In general, E_k represents the event in which E is true and there is kth order belief of E. If, at some state, there is belief of any order $k \in \mathbb{N}$ of event E, then E is said to be *commonly believed* at this state. That is to say, at this state, every player believes E, every player believes that every player believes E, and so on ad infinitum. Obviously, event E_* specified by

$$E_* := \bigcap_{k\in\mathbb{N}_0} E_k$$

represents the event where E is true and commonly believed among all players.

Remark 4.5 *Consider a framed strategic game $\mathbb{T}_\Gamma$ and let $E \subseteq \Omega$ be some event.*

(a) If $E := \prod_{i\in I} E^i$ is rectangular, then E_k is rectangular for every $k \in \mathbb{N}_0$ and E_* is rectangular.

(b) If $\mathbb{T}_\Gamma$ is regular and E is closed, then E_k is closed for every $k \in \mathbb{N}_0$ and E_* is closed.

In accordance with our above definition, event E is said to be true and commonly believed at state ω whenever $\omega \in E_*$ holds. Two alternative formal characterizations of such a state are given in the following remark. These characterizations are applied in the proofs of the next section.

Remark 4.6 *Consider a framed strategic game $\mathbb{T}_\Gamma$ and an event $E \subseteq \Omega$. Then the following statements are equivalent.*

(i) Event E is true and commonly believed at state ω.

(ii) There is an event $F \subseteq E$ having the properties $\omega \in F$ and $P^i(t^i_{\tilde{\omega}}) \subseteq Q^i_F(\tilde{\omega}^i)$ for every player $i \in I$ and every state $\tilde{\omega} \in F$.

(iii) It holds that $\omega \in E$ and $P^i(t^i_\omega) \subseteq Q^i_{E_*}(\omega^i)$ for every player $i \in I$,.

The purpose of framing a strategic game with a type space model is to view the game from the players' perspectives and to model their decision makings. As argued at the outset of this section, for those decision makers, games are decision problems under subjective uncertainty. More precisely, each player is faced with the problem of choosing one of her available strategies, given her subjective conjecture about the opponents' choices and conjectures. In Section 2, we introduced a general framework for describing decision problems. How the players' decision problems are contrived from a framed strategic game is addressed in the succeeding paragraphs.

According to our concepts put forward in Section 2, the configuration of a decision problem consists of a specification of the three attributes, namely, condition space, possibility space, and constraint. Let $\mathbb{T}_\Gamma$ be a framed strategic game and $i \in I$ be a player at state $\omega \in \Omega$, then the attributes of player i's decision problem are set up as follows. The condition space of player i at this state is determined by Ω^{-i}. It contains all conceivable profiles of strategy-type combinations of i's opponents. Hereafter, we refer to Ω^{-i} as the *basic uncertainty space* of i.[12] The *possibility set of player i at state ω* is given by $P^i(t^i_\omega)$ and is interpreted as the set of opponents' strategy-type combinations deemed possible by i. In order to specify the constraint of player i at state ω, we introduce mapping $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(s^i) := z^i(s^i; .) \circ \mathrm{proj}^{\Omega^{-i}}_{S^{-i}}$, which converts each strategy $s^i \in S^i$ of player i into a payoff profile on player i's uncertainty space Ω^{-i}. Payoff profiles on player i's uncertainty space are called *acts of player i*, and act $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(s^i)$ is termed the *act induced by strategy s^i*. Finally, the *constraint of player i at state ω* is specified by $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(S^i) := \{\mathfrak{a}^i_{\mathbb{T}_\Gamma}(s^i) : s^i \in S^i\}$.

Bringing together these specifications, the decision problem of some player i participating in game Γ and being in state ω is summarized by the tuple

$$\Phi^i_{\mathbb{T}_\Gamma(\omega)} := \big(P^i(t^i_\omega), \mathfrak{a}^i_{\mathbb{T}_\Gamma}(S^i)\big)_{\Omega^{-i}} \ ,$$

[12]This terminology deviates slightly from the one generally used in decision theory under subjective uncertainty and dates back to Savage (1954). He terms the uncertainty space of a decision maker as state space and the elements of this space as states of the world. The reason for this terminological deviation is that we study the decision problems of several decision makers who interact with each other (i.e., who play a game). A condition space of such a decision maker describes her uncertainty and lists the conceivable choices and beliefs of her opponents whereas, as defined earlier in this section, the term state space is reserved for the collection of the conceivable choices and beliefs of all players. Our distinction between uncertainty space and state space relies on our implicit assumption that each player knows her own choice and belief.

where Ω^{-i} constitutes her basic uncertainty, $P^i(t^i_\omega)$ the set of profiles of strategy-type combinations of her opponents that she deems possible at state ω, and $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(S^i)$ is the set of acts on her basic uncertainty Ω^{-i} available to her. We call tuple $\Phi^i_{\mathbb{T}_\Gamma(\omega)}$ the *strategic decision problem of player i at state ω*.

Suppose that the framed strategic game $\mathbb{T}_\Gamma$ is regular and consider some player $i \in I$ at some state $\omega \in \Omega$. Then i's possibility set $P^i(t^i_\omega)$ is closed in Ω^{-i}. Furthermore, mappings $\text{proj}^{\Omega^{-i}}_{S^{-i}}$ and, for any arbitrary $s^i \in S^i$, $z^i(s^i; .)$ are continuous. Hence, i's constraint $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(S^i)$ contains only continuous acts. As argued in Remark 2.4, her constraint $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(S^i)$ is also a compact in $\mathbf{B}(\Omega^{-i})$. To sum up, we have established that whenever a regular framed strategic game is assumed the strategic decision problem of any player at any state shows up as a regular decision problem under condition space Ω^{-i}.

From now on, we denote by

$$\mathscr{D}^i_{\mathbb{T}_\Gamma} := \{\Phi^i_{\mathbb{T}_\Gamma(\omega)} : \omega \in \Omega\}$$

the set of all strategic decision problems of player i in framed strategic game $\mathbb{T}_\Gamma$. Let $\mathcal{C}^i$ be some choice rule, then

$$[\mathcal{C}^i] := \left\{\omega \in \Omega : \mathfrak{a}^i_{\mathbb{T}_\Gamma}(s^i) \in \mathcal{C}^i(\Phi^i_{\mathbb{T}_\Gamma(\omega)})\right\}$$

determines the set of states in which player i behaves as if *she applies choice rule $\mathcal{C}^i$* to solve her strategic decision problem at this state.

Remark 4.7 *Consider a regular framed strategic game $\mathbb{T}_\Gamma$. If choice rule $\mathcal{C}^i$ is regular and monotone in $\mathscr{D}^i_{\mathbb{T}_\Gamma}$, then $[\mathcal{C}^i]$ is closed in Ω.*

Whenever a family $(\mathcal{C}^i)_{i\in I}$ of choice rules is supposed, we denote by $C := \bigcap_{i\in I}[\mathcal{C}^i]$ the statement saying that each player $i \in I$ applies choice rule $\mathcal{C}^i$. Analogous to our above notation, we stipulate $C_0 := C$ and denote by C_k the event in which C is true and there is kth order belief of C. Event C_* represents the event in which C is true and this is commonly believed. Due to Remark 4.7, event C is closed in Ω whenever all players stick to regular and monotone choice rules. Provided these assumptions on players' choices, Remark 4.5 implies that C_* is rectangular and closed in Ω. Since the latter result is used in the proofs of the next section, it is summarized in the following remark.

Remark 4.8 *Consider a regular framed strategic game $\mathbb{T}_\Gamma$ and a family $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ of choice rules. If each choice rule $\mathcal{C}^i$ is regular and monotone in $\mathscr{D}^i_{\mathbb{T}_\Gamma}$, then C_* is rectangular and closed.*

Consider a strategic game Γ and a family $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ of choice rules. A strategy profile $s \in S$ is said to be *possible* (or *justifiable*) *under choice rule $\mathcal{C}$ following behavior and common belief of the choice rule $\mathcal{C}$ following behavior* if there is a regular type space model $\mathbb{T}$ to Γ so that its state space Ω contains a state ω satisfying both $s_\omega = s$ and $\omega \in C_*$. Hereafter, we denote by $\text{CB}_{\mathcal{C}}$ the solution concept assigning to each strategic game the set of strategy profiles possible if players apply choice rules $\mathcal{C}$ and commonly believe it.

5 The Equivalence Result

In this section, we compare the two solution concepts $\mathcal{IA}_{\mathcal{C}}$ and $\mathcal{CB}_{\mathcal{C}}$, which have been introduced in Section 3 and 4, respectively. We aim to identify properties of players' choice rules $\mathcal{C}$ so that these two solution concepts coincide. The benefit of such a coincidence would be that the process of iterated application of choice rules obtains a pellucid epistemic underpinning. Obviously, it would mean that the solution obtained by iterated application of choice rules would give exactly the strategy profiles that are possible if players apply these choice rules and this is commonly believed among them. To accomplish such epistemic foundation another choice rule property is needed in addition to the properties rules already introduced in the previous sections.

Consider two regular decision problems, $(E, \mathfrak{A})_\Theta$ and $(\tilde{E}, \tilde{\mathfrak{A}})_{\tilde{\Theta}}$. The second is called an *extension* of the first if there exists a surjective, continuous, and closed mapping $\kappa : \tilde{\Theta} \to \Theta$ that satisfies $\kappa(\tilde{E}) = E$ and $\tilde{\mathfrak{A}} = \{\mathfrak{a} \circ \kappa : \mathfrak{a} \in \mathfrak{A}\}$. Evidently, an extension of a regular decision problem is a regular decision problem in which the conditions of the original decision problem are relabeled, or payoff equivalent conditions (i.e., conditions yielding the same payoffs as some other condition) are removed or added. Whenever a choice rule is immune to such modifications of regular decision problems, it is called independent of payoff equivalent states.

Definition 5.1 *Consider a system $\tilde{\mathscr{D}}_\Theta \subseteq \mathscr{D}_\Theta$ of decision problems under condition space Θ. A choice rule $\mathcal{C}$ is called independent of payoff equivalent conditions in $\tilde{\mathscr{D}}_\Theta$ if*

$$\mathcal{C}(\tilde{E}, \tilde{\mathfrak{A}})_{\tilde{\Theta}} = \{\mathfrak{a} \circ \kappa : \mathfrak{a} \in \mathcal{C}(E, \mathfrak{A})_\Theta\}$$

holds for every regular decision problem $(E, \mathfrak{A})_\Theta \in \tilde{\mathscr{D}}_\Theta$ and for every extension $(\tilde{E}, \tilde{\mathfrak{A}})_{\tilde{\Theta}}$ of it.

Without difficulty, the following remark can be verified.

Remark 5.2 *Consider some condition space Θ. The choice rules $\mathcal{MM}$, $\mathcal{PR}$, $\mathcal{SD}$, $\mathcal{SD}_+$, $\mathcal{SU}_m$, and $\mathcal{SU}_p$ are independent of payoff equivalent conditions in $\mathscr{D}_\Theta$.*

Our first lemma of this section states properties of choice rules implying that each strategy profile possible under choice rule following behavior and common belief of it survives the deletion process of iterated application of these choice rules.

Lemma 5.3 *Consider a regular strategic game Γ and a family $\mathcal{C} := (\mathcal{C}^i)_{i \in I}$ of choice rules. If each choice rule $\mathcal{C}^i$ is non-empty, closed, monotone, independent of payoff equivalent conditions, and satisfies Aizerman's property in $\mathscr{D}^i_\Gamma$, then $CB_{\mathcal{C}}(\Gamma) \subseteq IA_{\mathcal{C}}(\Gamma)$ holds.*

Proof. Consider a regular strategic game Γ and assume $s_* \in \mathrm{CB}_{\mathcal{C}}(\Gamma)$. Then there exists a regular framed strategic game $\mathbb{T}_\Gamma$ whose state space $\Omega = \prod_{i \in I}(S^i \times T^i)$ contains a state ω_* that satisfies $s_{\omega_*} = s$ and $\omega_* \in C_*$. It is known from Remark 4.8 that C_* is a rectangular and closed subset of Ω. Furthermore, as stated in Remark 4.2, projection proj_S from Ω into S is surjective, continuous, and closed. Consequently, $S_* := \mathrm{proj}_S C_*$ is a closed restriction of Γ. In the following, we establish that restriction S_* has the best choice property. For this purpose, pick an arbitrary player $i \in I$ and an arbitrary strategy $\tilde{s}^i \in S^i_*$. It follows from our construction of S_* that there is a state $\tilde{\omega} \in \Omega$ where both $\tilde{s}^i = s^i_{\tilde{\omega}}$ and $\tilde{\omega} \in C_*$ are satisfied. Since $C_* \subseteq C$ holds, $\tilde{s}^i \in \mathcal{C}^i(P^i(t^i_{\tilde{\omega}}), S^i)_{\Omega^{-i}}$ is valid. By Remark 4.2, projection $\mathrm{proj}^{\Omega^{-i}}_{S^{-i}}$ is surjective, continuous, and closed. Define $S^{-i}(t^i_{\tilde{\omega}}) := \mathrm{proj}^{\Omega^{-i}}_{S^{-i}}(P^i(t^i_{\tilde{\omega}}))$ and recall that

$$\mathfrak{a}^i_{\mathbb{T}_\Gamma}(s^i) := \mathfrak{a}^i_\Gamma(s^i) \circ \mathrm{proj}^{\Omega^{-i}}_{S^{-i}}$$

is given for every $s^i \in S^i$ where $\mathfrak{a}^i_\Gamma(s^i)$ denotes the payoff profile induced by strategy s^i on S^{-i} and $\mathfrak{a}^i_{\mathbb{T}_\Gamma}(s^i)$ the payoff profile induced by strategy s^i on Ω^{-i}. Obviously, strategic decision problem $(P^i(t^i_{\tilde{\omega}}), \mathfrak{a}^i_{\mathbb{T}_\Gamma}(S^i))_{\Omega^{-i}}$ of player i at state $\tilde{\omega}$ proves to be an extension of decision problem $(S^{-i}(t^i_{\tilde{\omega}}), \mathfrak{a}^i_\Gamma(S^i))_{S^{-i}}$. Since choice rule $\mathcal{C}^i$ is assumed to be independent of payoff equivalent states in $\mathscr{D}^i_\Gamma$, we obtain $\tilde{s}^i \in \mathcal{C}^i(S^{-i}(t^i_{\tilde{\omega}}), S^i)_{S^{-i}}$. Due to the equivalence of statements (i) and (iii) of Remark 4.6, $P^i(t^i_{\tilde{\omega}}) \subseteq Q^i_{C_*}(\tilde{\omega}^i) = C^{-i}_*$ applies and, thus,

$$S^{-i}(t^i_{\tilde{\omega}}) := \mathrm{proj}^{\Omega^{-i}}_{S^{-i}}\left(P^i(t^i_{\tilde{\omega}})\right) \subseteq \mathrm{proj}^{\Omega^{-i}}_{S^{-i}}(C^{-i}_*) = S^{-i}_*$$

is satisfied. Since S^{-i}_* is closed in S^{-i}, monotonicity implies $\tilde{s}^i \in \mathcal{C}^i(S^{-i}_*, S^i)_{S^{-i}}$. Clearly, the latter result holds for every $i \in I$ and for every $\tilde{s}^i \in S^i_*$ and thus the best choice property of S_* is established. Because players' choice rules $\mathcal{C}$ are assumed to satisfy non-emptiness, closedness, monotonicity and Aizerman's property, the assumptions of Lemma 3.9 are fulfilled. Consequently, each strategy profile of S_* survives the iterated application of choice rules $\mathcal{C}$ on Γ. To complete our proof, we must return strategy profile s_*, which is chosen at state ω_*. Because $s_* \in S_*$ has been assumed, $s_* \in \mathcal{IA}_\mathcal{C}(\Gamma)$ must hold. □

The next lemma deals with the converse of the previous lemma. More precisely, it identifies properties of choice rules so that each strategy profile surviving the iterated application of these choice rules is possible if players follow these choice rules and commonly believe it.

Lemma 5.4 *Consider a regular strategic game Γ and a family $\mathcal{C} := (\mathcal{C}^i)_{i \in I}$ of choice rules. If every choice rule $\mathcal{C}^i$ is regular, reflexive, and independent of payoff equivalent conditions in $\mathscr{D}^i_\Gamma$, then $CB_\mathcal{C}(\Gamma) \supseteq IA_\mathcal{C}(\Gamma)$ holds.*

Proof. Consider a strategic game Γ and assume $s \in R_\infty := \mathrm{IA}_\mathcal{C}(\Gamma)$. Demonstrating $s \in \mathrm{CB}_\mathcal{C}(\Gamma)$ requires to specify a regular framed strategic game $\mathbb{T}_\Gamma$ whose state space $\Omega := \prod_{i \in I}(S^i \times T^i)$ contains a state ω where $s_\omega = s$ and $\omega \in C_*$ hold. The type space model $\mathbb{T} := (T^i, P^i)_{i \in I}$ to Γ is constructed as follows. For every player $i \in I$, her type space is given by singleton $T^i := \{t^i\}$ and her possibility set by $P^i(t^i_\omega) := \prod_{j \in I \setminus \{i\}}(R^j_\infty \times \{t^j\})$. Obviously, for every $i \in I$, possibility correspondence P^i is non-empty and continuous. Moreover, due to Remark 3.2, it is also closed-valued. Consider the closed event $F := \prod_{i \in I} R^i_\infty \times \{t^i\}$ and pick some state $\tilde{\omega} := (\tilde{s}^i, t^i)_{i \in I} \in F$. By our construction, $\tilde{s}^i_{\tilde{\omega}} \in R^i_\infty$ holds. Furthermore, Lemma 3.5 implies $\tilde{s}^i_{\tilde{\omega}} \in \mathcal{C}^i(R^{-i}_\infty, S^i)_{S^{-i}}$. Without difficulty, it can be shown that the strategic decision problem $(P^i(t^i_{\tilde{\omega}}), S^i)_{\Omega^{-i}}$ of player i at state $\tilde{\omega}$ is an extension of decision problem $(R^{-i}_\infty, S^i)_{S^{-i}}$. By the independence of payoff equivalent conditions, $s^i_{\tilde{\omega}} \in \mathcal{C}^i(P^i(t^i_{\tilde{\omega}}), S^i)_{\Omega^{-i}}$ results. Because the latter statement holds for every state $\tilde{\omega} \in F$ and every player $i \in I$, we have established $F \subseteq C$. Furthermore, $P^i(t^i_{\tilde{\omega}}) \subseteq Q^i_F(\tilde{\omega}^i) = F^{-i}$ is satisfied for every state $\tilde{\omega} \in F$ and every player $i \in I$. Hence, due to Remark 4.6, players act according to choice rules $\mathcal{C}$ and this is commonly believed at every state belonging to F. In formal terms, $F \subseteq C_*$ has been proved. Finally, consider state $\omega := (s^i, t^i)_{i \in I}$, in which the strategy profile $s = s_\omega$ is realized. Since $s \in R_\infty$ has been presupposed, $\omega \in F$ holds and, hence, $\omega \in C_*$ results. Summing up our findings, it can be stated that our regular framed strategic game $\mathbb{T}_\Gamma$ contains a state ω in which both $s = s_\omega$ and $\omega \in C_*$ are satisfied. Briefly stated, $s \in \text{CB}_{\mathcal{C}}(\Gamma)$ is verified as desired. □

Putting the previous two lemmata together, we obtain following equivalence result regarding the solutions of $\text{IA}_{\mathcal{C}}$ and $\text{CB}_{\mathcal{C}}$.

Theorem 5.5 *Consider a regular strategic game Γ and a family $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ of choice rules. If each choice rule $\mathcal{C}^i$ is regular, reflexive, monotone, independent of payoff equivalent conditions, and satisfies Aizerman's property in $\mathscr{D}^i_\Gamma$, then $CB_{\mathcal{C}}(\Gamma) = IA_{\mathcal{C}}(\Gamma)$ holds.*

Theorem 5.5 is the central result of this paper and strict in the following sense. None of its conditions an be omitted without undermining the equivalence between the solution obtained by iterated application of choice rules and the solution resulting from choice rule following behavior and common belief of it. In Appendix A, we present four examples of choice rules - admittedly, some of them are very contrived - that confirm the strictness of Theorem 5.5.

Corollary 5.6 *Consider a family $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ of choice rules. If each choice rule $\mathcal{C}^i$ is regular, reflexive, monotone, independent of payoff equivalent conditions, and satisfies Aizerman's property in every system $\mathscr{D}^i_\Gamma$ of every regular strategic game Γ, then $CB_{\mathcal{C}}(\Gamma) = IA_{\mathcal{C}}(\Gamma)$ holds for every regular strategic game Γ.*

In order to concretize this corollary, let us return to the choice rules discussed in Secion 2. As demonstrated in our Remarks 2.3, 3.4 and 5.2, with the exception of the rules $\mathcal{MM}$ and $\mathcal{SD}$ these choice rules are regular, reflexive, and monotone, and satisfy both Aizerman's property and the independence of payoff equivalent conditions in the complete system of decision problems. Hence, by Corollary 5.6, we obtain the following result.

Remark 5.7 *Consider a family $\mathcal{C} := (\mathcal{C}^i)_{i\in I}$ of choice rules where $\mathcal{C}^i \in \{\mathcal{PR}, \mathcal{SD}_+, \mathcal{SU}_p, \mathcal{SU}_m\}$ holds for every $i \in I$. Then $CB_{\mathcal{C}}(\Gamma) = IA_{\mathcal{C}}(\Gamma)$ holds for every regular strategic game Γ.*

Remark 5.7 contains well-known results of epistemic game theory e.g., the epistemic characterization results of Mariotti (2003) and Chen et al. (2007). While the former author has demonstrated that the solution concept of iterated application of the point rationality rule is epistemically characterizable by choice-rule following behavior and common belief of it, the latter have demonstrated that such epistemic characterization is also valid for the solution concept of iterated application of the strict undominance rule.

Moreover, the above remark includes results that are not detectable in the preference-based framework of Epstein (1997). As argued at the end of Section 2, the modified strict dominance rule cannot be rationalized by any class of preference relations, and the characterizations results of Epstein (1997) are not applicable to this rule. In contrast, our framework provides an epistemic justification for the iterated application of this rule, namely, that according to the above remark, it is epistemically justifiable by the players' compliance with this rule and the common belief of it.

By means of Remark 5.7 and of the proposition of Zimper (2005), we are also able to substantially generalize the well-known epistemic characterization result by Brandenburger and Dekel (1987) and Tan and Werlang (1988). As already mentioned they have demonstrated that for the class of finite strategic games the solution concept of iterated strict undominance in mixed payoff profiles is characterizable by Bayesian rationality (i.e., maximizing expected payoff) and common belief of it. In order to show that this characterization also applies to metrizable regular strategic games, i.e., regular strategic games in which the players' strategy sets are metrizable, we first make use of Zimper's proposition. This generalizes Lemma 3 of Pearce (1984) and states that the equivalence between being Bayesian rational and following the choice rule of strict undominance in mixed payoff profiles holds even for the class of metrizable regular strategic games. Since according to Remark 5.7, the iterated application of this choice rule is characterizable by choice-rule following behavior and common belief of it for this class of strategic games, the desired generalization of the result by Brandenburger and Dekel (1987) and Tan and Werlang (1988) follows.[13]

As demonstrated in the decision problem of Figure 3 in Section 2, the maximin rule $\mathcal{MM}$ does not generally satisfy the property of reflexivity nor that of monotonicity. The consequences of these failures can be seen in our introductory example of strategic game Γ_1. For this the set of strategy profiles possible under maximin rule following behavior and common belief of it is neither a subset nor a superset of the set of strategy profiles surviving the iterated application of the maximin rule.

6 Discussion

This paper has dealt with the issue whether the solution of a strategic game originating from iterated application of choice rules coincides with the set of strategy profiles realized by players who follow these rules and commonly believe this. We have provided meaningful and general conditions on choice rules ensuring this coincidence. The issue of this paper is not new and has already been addressed by Epstein (1997) and Apt and Zvesper (2010). We have aimed to overcome the limitations of these papers. While Epstein (1997) considers only finite strategic games

[13]It is noteworthy that, whenever the strategic game is also concave-like, worthwhile alternative epistemic interpretations of the solution concept of iterated strict undominance in mixtures could be inferred. Such alternative interpretations result from the indistinguishability theorem by Chen and Luo (2012). A strategic game is said to be concave-like if for every player $i \in I$, for every strategies $s^i_*, s^i_{**} \in S^i$, and for every $\lambda \in [0,1]$ there is a strategy $s^i \in S_i$ so that $z^i(s^i, s^{-i}) \geq \lambda z^i(s^i_*, s^{-i}) + (1-\lambda) z^i(s^i_{**}, s^{-i})$ holds for every $s_{-i} \in S_{-i}$. A prominent example of a concave-like strategic game is the mixed extension of a finite strategic game. The indistinguishability theorem states that for concave-like and metrizable regular strategic games the solutions originating from iterated application of choice rules are identical whenever these rules are rationalizable by some preference model containing at least all preferences representable by some expected payoff function. To exemplify its substance, consider the following two preference-based choice rules. The first is induced by the preference model consisting of the preference relations representable by some expected payoff function, and the second is induced by the preference model consisting of the preference relations representable by some maximin expected payoff function (see Gilboa and Schmeidler, 1989). The indistinguishability theorem ensures that for every concave-like and metrizable regular strategic game the solutions obtained by the iterated application of the former rule correspond with those obtained by the iterated application of the latter rule. Finally, from this theorem we can infer alternative epistemic characterizations of the solution concept of iterated strict undominance in mixed payoff profiles for this class of strategic games. Indeed, this concept can be characterized by the compliance of any choice rule induced by some preference model containing at least all preference relations representable by some expected payoff function and the common belief of this compliance.

and preference-based choice rules, Apt and Zvesper (2010) define the process of iterated application of choice rules in a nonstandard way. Indeed, Apt and Zvesper (2010) make use of transfinite ordinals and apply the choice rules on the initial set of strategies (rather than on the remaining strategies as standard game theory does) in each round of the process.

Our approach has been, instead, to choose the following setup. Like Apt and Zvesper (2010), we have considered arbitrary strategic games and choice rules. However, in order to avoid the use of transfinite ordinals, we have endowed strategic games with a topological structure. More precisely, we have assumed that the player's strategy sets are compact Hausdorff and that their payoff functions are continuous on the set of possible strategy profiles, which, in turn, is endowed with the product topology. Since this kind of strategic games constituted our scope of application, we have given them a specific name and termed them regular.

Theorem 5.5 summarizes our work. It states four substantial assumptions on choice rules ensuring that the solution obtained by iterated application of these rules coincides with the set of strategy profiles realized by players who follow these rules and commonly believe this. More precisely, if the players' choices rules satisfy - besides the technical assumption of regularity - the properties of reflexivity, monotonicity, Aizerman's property, and the independence of payoff equivalent conditions, then this coincidence applies. This result proves to be strict in the following sense. As established in Appendix A, none of the four substantial properties can be omitted without eliminating the coincidence. We point out that the latter finding does not mean that the restrictions imposed in Theorem 5.5 are the weakest possible to ensure coincidence. It might be a worthwhile topic of future research to determine whether less demanding assumptions on the strategic game or the choice rules could be imposed without ruining its conclusion.

Even if the assumptions of Theorem 5.5 could be considerably weakened, it is in its current form nevertheless applicable to a wide range of strategic games and choice rules. Remark 5.7 lists prominent choice rules that satisfy the conditions presupposed in Theorem 5.5 in every regular strategic game. Hence, any solution concept originating from iterated application of such choice rules gives for every regular strategic game the strategy profiles that might be chosen by players who follow these rules and commonly believe this. Or put differently, such solution concepts are epistemically characterizable by choice-rule following behavior and common belief of it. We note that the list of choice rules compiled in Remark 5.7 is by no means exhaustive. It can be shown that other prominent choice rules like Börgers' inherent undominance rule (see Börgers, 1993) also satisfy the required properties in every regular strategic game.

Moreover, our findings are not only suitable for inferring general statements such as that in Remark 5.7. By means of the intermediate results leading to Theorem 5.5, we are also able to disclose significant relationships between the process of iterated application of choice rules and the epistemic assumption of choice-rule following behavior and common belief of it even for choice rules that do not satisfy all properties required in Theorem 5.5. Take, for example, the choice rule of weak undominance in pure payoff profiles or that of weak undominance in mixed payoff profiles. It turns out that both rules are not regular in general. More precisely, while they are non-empty and constraint continuous from above in every regular strategic game, they fail to be closed and possibility continuous from above in some (non-finite) regular strategic games. Furthermore, it could be established that both choice rules satisfy reflexivity, Aizerman's property, and independence of irrelevant conditions for every regular game but not necessarily monotonicity. Since the regularity assumption is trivially fulfilled for every finite strategic game, Lemma 5.4 is applicable to both choice rules at least in this subclass of strategic games. Hence, we infer that for every finite strategic game each strategy profile surviving the iterated application of one of these choice rules might be realized by players who follow this rule and commonly believe this. However, the converse of the latter statement is not generally true. As demonstrated in Example A.2 of Appendix A, there are finite strategic games in which these choice rules fail to be monotone and strategy profiles exist that do not survive the iterated application of one of these rules but nevertheless might be realized by players who follow this rule and commonly believe this.

Finally, it might be promising to study the converse of the issue addressed by us in this paper. Instead of examining which properties of choice rules ensure that the solution obtained by iterated application of choice rules is justifiable by choice-rule following behavior and common belief of it, one could also take this justification as given and ask what properties of choice rules are implied by it. We have not confronted this question in this paper, but it might be worthwhile to address it in future research.

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Appendix A - Examples regarding the Strictness of Theorem 5.5

In the following examples, choice rules are considered that satisfy all but one of the properties listed in Theorem 5.5. The property of reflexivity is violated in the first example, that of monotonicity in the second example, Aizerman's property in the third example, and that of independence of payoff-equivalent conditions in the fourth example.

Example A.1 Let $\mathcal{ND}_2$ be the choice rule that considers unfavorable any available act that is strictly dominated on the possibility set by some close available act. The closeness between two acts is measured by the uniform metric where, however, only the conditions of the possibility set are considered. Two acts are said to be close whenever the closeness between them is less than 2. In formal terms, choice rule $\mathcal{ND}_2$ is specified by

$$\mathcal{ND}_2(P, \mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A} : \text{there is no } \mathfrak{b} \in \mathfrak{A} \text{ so that } \mathfrak{b}_\theta > \mathfrak{a}_\theta \text{ holds for every } \theta \in P \text{ and } ||\mathfrak{b}_P - \mathfrak{a}_P||_\infty < 2\}$$

for every decision problem $(P, \mathfrak{A})_\Theta$ of any condition space Θ where $||.||_\infty$ denotes the uniform norm on $\mathbb{R}^P$. Suppose the regular and finite strategic game Γ depicted below is solved by iterated application of choice rule $\mathcal{ND}_2$. Obviously, in the first round strategy d is deleted and in the second round strategy r is deleted. After that the deletion process stops and, thus, strategy profiles (u, l) and (m, l) constitute the solution of this process.

Player C

Player R		l	r
	u	$(2, 1)$	$(3, 0)$
	m	$(0, 1)$	$(0, 0)$
	d	$(1, 0)$	$(2, 2)$

Figure 4: Strategic game Γ_2

Without difficulty, it can be seen that choice rule $\mathcal{ND}_2$ violates the property of reflexivity in system $\mathscr{D}^R_{\Gamma_2}$ of possible decision problems of player R in strategic game Γ_2. However, for each system $\mathscr{D}^i_{\Gamma_2}$ of each player $i \in \{C, R\}$, this choice rule satisfies the other properties stated in Theorem 5.5. In the following, we establish that some of the strategy profiles surviving the iterated application of choice rule $\mathcal{ND}_2$ would not be chosen by C and R if both follow this choice rule and commonly believe it.

To this end, frame strategic game Γ_2 by some regular type space model $\mathbb{T} := (T^i, P^i)_{i \in \{C,R\}}$ which contains a state ω where both players follow choice rule $\mathcal{ND}_2$ and commonly believe it. In the following we show by contradiction that strategy profile (m, l) is not chosen at state ω. Suppose on the contrary that R chooses strategy m at state ω. Let $\tilde{S}^C := \text{proj}^{\Omega^C}_{S^C} P^R(t^R_\omega)$ be the set of C's strategies that R considers possible at state ω. Obviously, if $\tilde{S}^C = \{l\}$, then R's choice is inconsistent in following choice rule $\mathcal{ND}_2$. let us turn to the remaining case $r \in \tilde{S}^C$. Because player R believes at state ω that player C follows choice rule $\mathcal{ND}_2$, player R must also believe at state ω that player C considers it possible that she could choose d. However, this belief conflicts with her belief that he believes that she follows choice rule $\mathcal{ND}_2$ and, thus, that he believes that she does not choose d. Thereby we have established that strategy profile (m, l), which survives the iterated application of choice rule $\mathcal{ND}_2$, is not chosen by players who comply with choice rule $\mathcal{ND}_2$ and commonly believe it. By Lemma 5.3, strategy profile (u, l) proves to be the only strategy profile of Γ_2, which could be realized by such players.

Example A.2 Consider the choice rule $\mathcal{WU}_p$ of weak undominance, which favors for every decision problem the available acts which are weakly undominated on the possibility set. The formal specification of this choice is provided in Appendix Appendix C. Suppose the regular and finite strategic game Γ_3 depicted below is solved by iterated application of the weak undominance rule. Obviously, only strategy profile (u, l) survives this deletion process.

Player C

		l	r
Player R	u	$(1,1)$	$(0,0)$
	d	$(0,0)$	$(0,0)$

Figure 5: Strategic game Γ_3

It turns out that choice rule $\mathcal{WU}$ satisfies all properties listed in Theorem 5.5 for every system $\mathscr{D}^i_{\Gamma_3}$ of strategic decision problems of every player $i \in \{C, R\}$ with the exception of the property of monotonicity. Consequently, Lemma 5.4 implies that every strategy profile of Γ_3 surviving the iterated application of the weak undominance rules could be chosen by players who follow this choice rules and commonly believe this. However, as shown next, strategy profile (d, l) is not the only one that could be realized by such players.

To see this, supplement strategic game Γ_3 with the regular type space model $(T^i, P^i)_{i\in\{C,R\}}$ where $T^i := \{t^i\}$ for each player $i \in \{C, R\}$, $P^C(t^C) := \{d\}\times T^R$ and $P^R(t^R) := \{r\}\times T^C$ are given. Obviously, both players act at state $\omega := (d, t^R, r, t^C)$ as if they apply the weak undominance rule and, by Remark 4.6, there is also common belief of such choice rule following behavior at this state. Hence, although strategy profile (d, r) does not survive the iterated application of the weak undominance choice rule, it could be realized by players who follow this rule and commonly believe it.

Example A.3 Consider the choice rule $\mathcal{AV}$ that favors the available acts yielding at least, at one condition considered possible, a payoff equal or larger than the average of the highest and lowest payoff realizable at this condition. In the case that, at some condition considered possible, a highest or a lowest payoff does not exist, every available act is approved by this rule. Formally, this choice rule is specified

$$\mathcal{AV}(P, \mathfrak{A})_\Theta := \Big\{ \mathfrak{a} \in \mathfrak{A} : \text{there exists some } \theta \in P \text{ so that, if } \sup_{b\in\mathfrak{A}} b_\theta < +\infty \text{ and} \\ \inf_{b\in\mathfrak{A}} b_\theta > -\infty, \text{ then } \mathfrak{a}_\theta \geq \frac{1}{2}\left(\sup_{b\in\mathfrak{A}} b_\theta + \inf_{b\in\mathfrak{A}} b_\theta \text{ holds} \right) \Big\}$$

for every decision problem $(P, \mathfrak{A})_\Theta$ of any condition space Θ. Suppose the regular and finite strategic game Γ_4 depicted below is solved by iterated application of this choice rule. Obviously, in the first round strategy d is deleted, in the second round strategy m and, finally, in the third round strategy r. Hence, the iterated application of this choice rule results in strategy profile (u, l).

		Player C	
		l	r
	u	$(3, 1)$	$(3, 0)$
Player R	m	$(2, 0)$	$(2, 1)$
	d	$(0, 1)$	$(0, 1)$

Figure 6: Strategic game Γ_4

Without difficulty, it can be established that choice rule $\mathcal{AV}$ violates Aizerman's property in system $\mathscr{D}^R_{\Gamma_4}$ but satisfies all other properties stated in Theorem 5.5 for each system $\mathscr{D}^i_{\Gamma_4}$ of each player $i \in \{C, R\}$. Consequently, by Lemma 5.4, strategy profile (u, l) could be realized by players who follow this choice rule and commonly believe it. However, as demonstrated next, a strategy profile different to (u, l) could also be chosen by such players.

To see this, frame game Γ by the regular type space model $(T^i, P^i)_{i \in \{C,R\}}$, where $T^i := \{t^i\}$ for each player $i \in \{C, R\}$, $P^C(t^C) := \{m\} \times T^R$ and $P^R(t^R) := \{r\} \times T^C$ are given. Obviously, both players act at state $\omega := (m, t^R, r, t^C)$ as if they follow choice rule $\mathcal{AV}$. Since $P^i(t^i_\omega) = \{\omega^i\}$ holds for every $i \in \{C, R\}$, Remark 4.6 implies that this choice rule following behavior is also commonly believed among them at state ω. Hence, we have found a strategy profile of Γ, namely (m, r), which does not survive the iterated application of choice rule $\mathcal{AV}$, but which could occur if both players follow this choice rule and commonly believe it.

Example A.4 Consider choice rule $\mathcal{LU}_{13}$, which selects the available acts which are strictly undominated on the possibility set or which yield the number 13 at two or more conditions considered possible. Such choice rule could be interpreted as decision maker's delight in lucky number 13. In formal terms, this rule is specified by

$$\mathcal{LU}_{13}(P, \mathfrak{A})_\Theta := \mathcal{SU}(P, \mathfrak{A})_\Theta \cup \{\mathfrak{a} \in \mathfrak{A} : \#\{\theta \in P : \mathfrak{a}_\theta = 13\} > 1\}$$

for every decision problem $(P, \mathfrak{A})_\Theta$ of every condition space Θ. Suppose the regular and finite strategic game Γ_5 depicted below is solved by iterated application of choice rule $\mathcal{LU}_{13}$. Obviously, in the first round strategy d is deleted and in the second round strategy r is deleted. Then the deletion process stops and, thus, only strategy profile (u, l) survives.

		Player C	
		l	r
Player R	u	$(14, 1)$	$(1, 0)$
	d	$(13, 1)$	$(0, 1)$

Figure 7: Strategic game Γ_5

As can be easily checked, choice rule $\mathcal{LU}_{13}$ satisfies all properties listed in Theorem 5.5 for every system $\mathscr{D}^i_{\Gamma_5}$ of every player $i \in N$ with the exception of the independence of payoff-equivalent condition. The violation of the latter property entails that, although strategy profile (d, l) is deleted in the first deletion round, it could be nevertheless chosen by players who follow this rule and commonly believe such choice rule following behavior.

To see this, consider the regular type space model $(T^i, P^i)_{i\in\{C,R\}}$ to Γ_5 whose type spaces are specified by $T^C := \{\tilde{t}^C, \hat{t}^C\}$ and $T^R = \{t^R\}$ and whose possibility correspondences are specified by $P^C(\tilde{t}^C) := \{u\} \times T^R$, $P^C(\hat{t}^C) := \{d\} \times T^R$ and $P^R(t^R) := \{l\} \times T^C$. Consider state $\omega := (d, t^R, l, \tilde{t}^C)$ and event $E := \{u, d\} \times T^R \times \{l\} \times T^C$. Obviously, $P^C(t^R_{\tilde{\omega}}) \subseteq Q^R_E(\tilde{\omega}^R)$ and $P^C(t^C_{\tilde{\omega}}) \subseteq Q^R_E(\tilde{\omega}^R)$ hold for every $\tilde{\omega} \in E$. It follows from Remark 4.6 that event E is true and commonly believed at state ω. From the latter result, we infer that, although strategy profile (d, l) does not survive the iterated application of choice rule $\mathcal{LU}_{13}$, it could occur if both players follow this choice rule and commonly believe it.

Appendix B - Proofs of Remarks

Proof of Remark 2.3

Before checking the choice rules for regularity, we make two helpful remarks. Consider a decision problem $(P, \mathfrak{A}) \in \mathscr{D}_\Theta$. From now on, we suppose that subspaces $\mathfrak{A} \subseteq \mathbf{B}(\Theta)$ and $P \subseteq \Theta$ are endowed with the relative topologies and space $\mathfrak{A} \times P$ is endowed with the product topology. Furthermore, z represents the mapping assigning to every pair $(\mathfrak{a}, \theta) \in \mathfrak{A} \times P$ the real number $z(\mathfrak{a}, \theta) := \mathfrak{a}_\theta$.

Remark B.1 *Consider a decision problem $(P, \mathfrak{A}) \in \mathscr{D}_\Theta$ where Θ is compact Hausdorff and $\mathfrak{A}$ is a subset of $\mathbf{CB}(\Theta)$. Then mapping z is continuous.*

Proof. Consider a net $(\mathfrak{a}^k, \theta^k)_{k\in K}$ in $\mathfrak{A} \times P$ converging to some point $(\mathfrak{a}, \theta) \in \mathfrak{A} \times P$. Because $\mathfrak{A} \times \Theta$ is endowed with the product topology, $(\mathfrak{a}_k)_{k\in K}$ converges (uniformly) in $\mathfrak{A}$ to $\mathfrak{a}$ and $(\theta_k)_{k\in K}$ converges in P to θ. Pick some arbitrary $\epsilon > 0$. Then there exists a $k_1 \in K$ so that $|\mathfrak{a}_k(\tilde{\theta}) - \mathfrak{a}(\tilde{\theta})| < \frac{\epsilon}{2}$ is satisfied for every $\tilde{\theta} \in P$ and for every $k \geq k_1$. Furthermore, by the continuity of payoff profile $\mathfrak{a}$, there exists $k_2 \in K$ so that $|\mathfrak{a}(\theta_k) - \mathfrak{a}(\theta)| < \frac{\epsilon}{2}$ is satisfied for every $k \geq k_2$. Define $k_0 := \max\{k_1, k_2\}$. Putting the last two inequalities together, we obtain

$$|z(\mathfrak{a}_k, \theta_k) - z(\mathfrak{a}, \theta)| = |\mathfrak{a}_k(\theta_k) - \mathfrak{a}(\theta)| \leq |\mathfrak{a}_k(\theta_k) - \mathfrak{a}(\theta_k)| + |\mathfrak{a}(\theta_k) - \mathfrak{a}(\theta)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for every $k \geq k_0$. Thus, the continuity of z is established. □

Remark B.2 *Consider a decision problem $(P, \mathfrak{A}) \in \mathscr{D}_\Theta$ where Θ is compact Hausdorff, P is closed in Θ, and $\mathfrak{A}$ is compact in $\mathbf{B}(\Theta)$. If mapping $e : \mathfrak{A} \times P \to \mathbb{R}$ is continuous, then*

$$\mathfrak{A}^P_e := \mathrm{proj}^{\mathfrak{A}\times P}_{\mathfrak{A}}\{(\mathfrak{a}, \theta) \in \mathfrak{A} \times P : e(\mathfrak{a}, \theta) \geq 0\}$$

is closed in $\mathbf{B}(\Theta)$.

Proof. Since mapping e is assumed to be continuous, $\{(\mathfrak{a}, \theta) \in \mathfrak{A} \times P : e(\mathfrak{a}, \theta) \geq 0\}$ is a closed subset of the topological space $\mathfrak{A} \times P$. Since Θ is assumed to be compact, P is also compact. Then

Tychonoff's Theorem implies the compactness of $\mathfrak{A} \times P$. Hence, $\{(\mathfrak{a}, \theta) \in \mathfrak{A} \times P : e(\mathfrak{a}, \theta) \geq 0\}$ is a compact subset of $\mathfrak{A} \times P$. Since $\mathfrak{A} \times P$ is endowed with the product topology, projection $\mathrm{proj}_{\mathfrak{A}}^{\mathfrak{A}\times \mathrm{P}}$ proves to be a continuous mapping. Thus, $\mathfrak{A}_e^P$ is a compact subset of $\mathfrak{A}$. Because $\mathfrak{A}$ is Hausdorff, $\mathfrak{A}_e^P$ is closed in $\mathfrak{A}$. Since constraint $\mathfrak{A}$ is closed in $\mathbf{B}(\Theta)$, $\mathfrak{A}_e^P$ is also closed in topological space $\mathbf{B}(\Theta)$. □

Remark B.3 *Consider some net $(P_k, \mathfrak{A})_{k\in K}$ of regular decision problems in $\mathscr{D}_\Theta$ and some regular decision problem $(P, \mathfrak{A}) \in \mathscr{D}_\Theta$ where Θ is compact Hausdorff and $P_k \searrow P$ holds. If a net $(\theta_k)_{k\in K}$ in Θ satisfies $\theta_k \in P_k$ for every $k \in K$, then $\emptyset \neq \mathrm{Lim}\{\theta_k : k \in K\} \subseteq P$ holds.*[14]

Proof. Since Θ is assumed to be compact, net $(\theta_k)_{k\in K}$ has a limit point. Pick some $\theta \in \mathrm{Lim}\{\theta_k : k \in K\}$. This means, net $(\theta_k)_{k\in K}$ contains a subnet which converges to θ. Without loss of generality, assume that $(\theta_k)_{k\in K}$ itself converges to this point. In the following, we prove that $\theta \in P$ must hold. Suppose the contrary is be true. Then there exists $k_0 \in K$ so that $\theta \notin P_k$ is satisfied for every $k \geq k_0$. Because Θ is compact (thus, P_{k_0} is compact) and Hausdorff, there are disjoint open subsets E and F of Θ having properties $P_{k_0} \subseteq E$ and $\theta \in F$. Note that, by assumption, $\theta_k \in P_{k_0}$ holds for every $k \geq k_0$. Hence, θ is not a limit point of net $(\theta_k)_{k\in K}$. However, this result is at odds with our above assumption about θ. Thus, we conclude that $\theta \in P$ holds. □

With the help of the three remarks above, we check the choice rules - one after the other - for the properties listed in Definition 2.2.

• Strict undominance in pure payoff profiles

(*Non-emptiness*) Pick some $\theta \in P$. According to Remark B.1, mapping z is continuous. Since constraint $\mathfrak{A}$ is assumed to be compact, the Weierstrass Theorem guarantees the existence of a payoff profile $\mathfrak{a} \in \mathfrak{A}$ so that $z(a, \theta) = \sup_{\mathfrak{b}\in\mathfrak{A}} z(\mathfrak{b}, \theta)$ holds. Consequently, $\mathfrak{a} \in \mathcal{SU}_p(P, \mathfrak{A})_\Theta$ is established.

(*Closedness*) For every $\mathfrak{b} \in \mathfrak{A}$, define mapping $e_\mathfrak{b} : \mathfrak{A} \times P \to \mathbb{R}$ where $e_\mathfrak{b}(\mathfrak{a}, \theta) := z(\mathfrak{a}, \theta) - z(\mathfrak{b}, \theta)$ is given for every $\theta \in P$ and $\mathfrak{a} \in \mathfrak{A}$. Due to Remark B.1, mapping $e_\mathfrak{b}$ is continuous for every $b \in \mathfrak{A}$. Note that

$$\begin{aligned}
\mathcal{SU}_p(P, \mathfrak{A})_\Theta &= \{\mathfrak{a} \in \mathfrak{A} : \text{ for every } \mathfrak{b} \in \mathfrak{A}, \text{ there is some } \theta \in P \text{ so that } \mathfrak{a}_\theta \geq \mathfrak{b}_\theta\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } z(\mathfrak{a}, \theta) \geq z(\mathfrak{b}, \theta)\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } e_\mathfrak{b}(\mathfrak{a}, \theta) \geq 0\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \mathrm{proj}_{\mathfrak{A}}^{\mathfrak{A}\times P}\{(\mathfrak{a}, \theta) \in \mathfrak{A} \times P : e_\mathfrak{b}(\mathfrak{a}, \theta) \geq 0\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \mathfrak{A}_{e_\mathfrak{b}}^P
\end{aligned}$$

holds. Due to Remark B.2, $\mathfrak{A}_{e_\mathfrak{b}}$ is closed in $\mathbf{B}(\Theta)$ for every $\mathfrak{b} \in \mathfrak{A}$. Thus, their intersection is also closed in $\mathbf{B}(\Theta)$.

[14]Let X be a topological space and $Y \subseteq X$. As usual, $\mathrm{Lim}Y$ denotes the set of all limit points of Y in X.

(*Continuity from above*) Consider some act $\mathfrak{a} \in \mathfrak{A}$ satisfying $\mathfrak{a} \in \cap_{k\in K}\mathcal{SU}_p(P_k, \mathfrak{A})_\Theta$. Pick some arbitrary payoff profile $\mathfrak{b} \in \mathfrak{A}$. Then, for every $k \in K$ there exists $\theta_k \in P_k$ so that $z(\mathfrak{a}, \theta_k) \geq z(\mathfrak{b}, \theta_k)$ holds. Due to Remark B.3, net $(\theta_k)_{k\in K}$ contains a subnet converging to some $\theta \in P$. Without loss of generality, we assume that $(\theta_k)_{k\in K}$ converges to this point. As stated in Remark B.1, mapping z is continuous and thus, from $z(\mathfrak{a}, \theta_k) \geq z(\mathfrak{b}, \theta_k)$ for every $k \in K$, it follows

$$\mathfrak{a}_\theta = \lim_k z(\mathfrak{a}, \theta_k) \geq \lim_k z(\mathfrak{b}, \theta_k) = \mathfrak{b}_\theta \ .$$

Since $\mathfrak{b}$ has been arbitrarily chosen, $\mathfrak{a} \in \mathcal{SU}_p(P, \mathfrak{A})_\Theta$ results. Thus, we have shown that choice rule $\mathcal{SU}_p$ is possibility set continuous from above. The proof that this choice rule is also constraint continuous from above is trivial.

• **Strict undominance in mixed payoff profiles**

(*Non-emptiness*) As can be easily verified, payoff profile $\mathfrak{a}$ of the above non-emptiness proof of choice rule $\mathcal{SU}_p$ is also favorable according to choice rule $\mathcal{SU}_m$.

(*Closedness*) Pick some probability measure $\mu \in \Delta(\mathfrak{A})$ and define mapping $z_\mu : \Theta \to \mathbb{R}$ by $z_\mu(\theta) := \int_{\mathfrak{A}} z(b, \theta) d\mu$. Obviously, z_μ is the expected payoff at state ω if the available strategies are randomly selected according to probability measure μ. Next, we establish that expected value mapping z_μ is continuous. For this purpose, consider net $(\theta_k)_{k\in K}$ in Θ converging to some $\theta \in \Theta$. Recall that constraint $\mathfrak{A}$ is assumed to be compact. By the Theorem of Arzela-Ascoli $\mathfrak{A}$ is equicontinuous. That means, there exists an open set U_θ containing point θ so that $|\mathfrak{b}_{\tilde{\theta}} - \mathfrak{b}_\theta| < \epsilon$ holds for every $\tilde{\theta} \in \Theta$ and for every $\mathfrak{b} \in \mathfrak{A}$. Hence, there exists $k_0 \in K$ so that

$$|z(\mathfrak{b}, \theta_k) - z(\mathfrak{b}, \theta)| < \epsilon$$

holds for every $k \geq k_0$ and for every $\mathfrak{b} \in \mathfrak{A}$. Consequently, we obtain for every $k \geq k_0$

$$\begin{aligned} |z_\mu(\theta_k) - z_\mu(\theta)| &= \left| \int_{\mathfrak{A}} z(\mathfrak{b}, \theta_k) d\mu - \int_{\mathfrak{A}} z(\mathfrak{b}, \theta) d\mu \right| \\ &= \left| \int_{\mathfrak{A}} \left(z(\mathfrak{b}, \theta_k) - z(\mathfrak{b}, \theta) \right) d\mu \right| \\ &\leq \int_{\mathfrak{A}} |z(\mathfrak{b}, \theta_k) - z(\mathfrak{b}, \theta)| \, d\mu \\ &< \int_{\mathfrak{A}} \epsilon \, d\mu = \epsilon \ . \end{aligned}$$

Thus, it is shown that expected value mapping z_μ is continuous. Consequently, the mapping $e_\mu : \mathfrak{A} \times P$ defined by

$$e_\mu(\mathfrak{a}, \theta) := z(\mathfrak{a}, \theta) - z_\mu(\theta)$$

for every $\mathfrak{a} \in \mathfrak{A}$ and for every $\theta \in P$ is continuous. Note that

$$\begin{aligned} \mathcal{SU}_m(P, \mathfrak{A})_\Theta &= \left\{ \mathfrak{a} \in \mathfrak{A} : \text{ for every } \mu \in \Delta(\mathfrak{A}) \text{ there is some } \theta \in P \text{ so that } \mathfrak{a}_\theta \geq \int_{\mathfrak{A}} \mathfrak{b}_\theta d\mu \right\} \\ &= \bigcap_{\mu \in \Delta\mathfrak{A}} \{ \mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } z(\mathfrak{a}, \theta) \geq z_\mu(\theta) \} \\ &= \bigcap_{\mu \in \Delta\mathfrak{A}} \text{proj}_{\mathfrak{A}}^{P\times\mathfrak{A}} \{ (\mathfrak{a}, \theta) \in \mathfrak{A} \times P : \text{ there is some } \theta \in P \text{ so that } e_\mu(\mathfrak{a}, \theta) \geq 0 \} \\ &= \bigcap_{\mu \in \Delta\mathfrak{A}} \mathfrak{A}^p_{e_\mu} \end{aligned}$$

holds. By Remark B.2, $\mathfrak{A}^{p}_{e_\mu}$ is closed in $\mathbb{B}(\Theta)$ for every $\mu \in \Delta(\mathfrak{A})$. Hence, their intersection is also closed in $\mathbb{B}(\Theta)$.

(*Continuity from above*) For every $\mu \in \Delta(\mathfrak{A})$, let e_μ be the mapping introduced in the preceding proof of closedness (i.e., $e_\mu(\mathfrak{a}, \theta) := z(\mathfrak{a}, \theta) - \int_{\mathfrak{A}} z(b, \theta) d\mu$ is given for every $\theta \in P$ and for every $\mathfrak{a} \in \mathfrak{A}$). As argued there, e_μ is continuous for every $\mu \in \Delta(\mathfrak{A})$. Consider payoff profile $\mathfrak{a} \in \mathfrak{A}$ which satisfies $\mathfrak{a} \in \cap_{k\in K} \mathcal{SU}_m(P_k, \mathfrak{A})_\Theta$. Pick some $\mu \in \Delta(\mathfrak{A})$. Obviously, for every $k \in K$ there exists $\theta_k \in P_k$ so that $e_\mu(\mathfrak{a}, \theta_k) \geq 0$ holds. By Remark B.3 net $(\theta_k)_{k\in K}$ contains a subnet converging to some $\theta \in P$. Without loss of generality, we assume that $(\theta_k)_{k\in K}$ converges to this point. Due to the continuity of e_μ we obtain $e_\mu(\mathfrak{a}, \theta) \geq 0$. Note that the latter result says nothing but $\mathfrak{a}_\theta \geq \int_{\mathfrak{A}} \mathfrak{b}_\theta d\mu$. Because the latter inequality holds for every $\mu \in \Delta(\mathfrak{A})$, we attain $\mathfrak{a} \in \mathcal{SU}_m(P, \mathfrak{A})_\Theta$. The proof that choice rule $\mathcal{SU}_m$ is also constraint continuous from above is trivial.

• **Point rationality**

(*Non-emptiness*) As can be easily verified, payoff profile $\mathfrak{a}$ in the above non-emptiness proof of choice rule $\mathcal{SU}_p$ is also favorable according to choice rule $\mathcal{PR}$.

(*Closedness*) Let $z_{\sup} : \Theta \to \mathbb{R}$ be the mapping where $z_{\sup}(\theta) := \sup_{\mathfrak{a}\in\mathfrak{A}} z(\mathfrak{a}, \theta)$ holds for every $\theta \in \Theta$. By assumption $\mathfrak{A}$ is compact in $\mathbb{B}(\Theta)$, and by Remark B.1 mapping z is continuous. Thus, standard arguments imply that mapping $z_{\sup}$ is continuous. Next, define mapping $e : \mathfrak{A} \times P \to \mathbb{R}$ by $e(\mathfrak{a}, \theta) := z(\mathfrak{a}, \theta) - z_{\sup}(\theta)$ for every $\theta \in P$ and for every $\mathfrak{a} \in \mathfrak{A}$. By Remark B.1 and the continuity of $z_{\sup}$, mapping e turns out to be continuous. Because

$$\begin{aligned}
\mathcal{PR}(P, \mathfrak{A})_\Theta &= \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } \mathfrak{a}_\theta \geq \mathfrak{b}_\theta \text{ holds for every } \mathfrak{b} \in \mathfrak{A}\} \\
&= \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } z(\mathfrak{a}, \theta) \geq \sup_{\mathfrak{b}\in\mathfrak{A}} z(\mathfrak{b}, \theta)\} \\
&= \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } z(\mathfrak{a}, \theta) \geq z_{\sup}(\theta)\} \\
&= \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } e(\mathfrak{a}, \theta) \geq 0\} \\
&= \mathrm{proj}^{\mathfrak{A}\times\Theta}_{\mathfrak{A}} \{(\mathfrak{a}, \theta) \in \mathfrak{A} \times P : e(\mathfrak{a}, \theta) \geq 0\} \\
&= \mathfrak{A}^P_e
\end{aligned}$$

holds, we conclude from Remark B.2 that $\mathfrak{A}^P_e$ is closed in $\mathbb{B}(\Theta)$.

(*Continuity from above*) Let e be the mapping introduced in the preceding proof (i.e., $e(\mathfrak{a}, \theta) := z(\mathfrak{a}, \theta) - \sup_{\mathfrak{b}\in\mathfrak{A}} z(\mathfrak{b}, \theta)$ is given for every $\theta \in P$ and for every $\mathfrak{a} \in \mathfrak{A}$). As argued there, e is continuous. Consider payoff profile $\mathfrak{a} \in \mathfrak{A}$ which satisfies $\mathfrak{a} \in \cap_{k\in K} \mathcal{PR}(P_k, \mathfrak{A})_\Theta$. This means that for every $k \in K$ there exists $\theta_k \in P_k$ so that $e(\mathfrak{a}, \theta_k) \geq 0$ holds. By Remark B.3 net $(\theta_k)_{k\in K}$ contains a subnet converging to some $\theta \in P$. Without loss of generality, we assume that $(\theta_k)_{k\in K}$ converges to this point. The continuity of e guarantees that $e(\mathfrak{a}, \theta) \geq 0$ is satisfied. Note that the latter result says nothing but that $\mathfrak{a}_\theta \geq \mathfrak{b}_\theta$ holds for every $\mathfrak{b} \in \mathfrak{A}$. Therefore, $\mathfrak{a} \in \mathcal{PR}(P, \mathfrak{A})_\Theta$ is established. The proof that choice rule $\mathcal{PR}$ is also constraint continuous from above is trivial.

• **Strict dominance**

(*Non-emptiness*) The choice rule $\mathcal{SD}$ violates the property of non-emptiness as can be easily seen by following trivial finite and regular strategic game Γ which is known as the matching pennies game.

		Player C	
		l	r
Player R	u	$(1,0)$	$(0,1)$
	d	$(0,1)$	$(1,0)$

Figure 8: Strategic game Γ_6

(*Closedness*) There are two mutually exclusive cases. Either is choice set $\mathcal{SD}(P,\mathfrak{A})_\Theta$ empty or choice set $\mathcal{SD}(P,\mathfrak{A})_\Theta$ is a singleton. Note that any singleton of a Hausdorff space is closed. Therefore choice set $\mathcal{SD}(P,\mathfrak{A})_\Theta$ is closed in both cases.

(*Continuity from above*) Suppose $\mathfrak{a} \in \cap_{k\in K}\mathcal{SD}(P_k,\mathfrak{A})_\Theta$ hold. Pick some $k \in K$. By assumption, $\mathfrak{a}_\theta > \mathfrak{b}_\theta$ holds for every $\theta \in P \subseteq P_k$. Consequently, $\mathfrak{a} \in \cap_{k\in K}\mathcal{SD}(P,\mathfrak{A})_\Theta$ applies. To show that choice rule $\mathcal{SD}_+$ is also constraint continuous from above is trivial.

• Modified strict dominance

(*Non-emptiness*) Non-emptiness follows immediately from the definition of this choice rule.

(*Closedness*) There are two mutually exclusive cases. Either is set $\mathcal{SD}(P,\mathfrak{A})_\Theta$ empty or $\mathcal{SD}(P,\mathfrak{A})_\Theta$ is non-empty. In the former case $\mathcal{SD}_+(P,\mathfrak{A})_\Theta$ corresponds to $\mathfrak{A}$ and, since $\mathfrak{A}$ is assumed to be a compact subset of Hausdorff space $\mathbb{B}(\Theta)$, is closed in $\mathbb{B}(\Theta)$. In the other case $\mathcal{SD}(P,\mathfrak{A})_\Theta$ is a singleton. Because any singleton of a Hausdorff space is closed, we also obtain the desired result even for this case.

(*Continuity from above*) Suppose $\mathfrak{a} \in \cap_{k\in K}\mathcal{SD}_+(P_k,\mathfrak{A})_\Theta$ hold. If there is $k_0 \in K$ so that $\mathfrak{a} \in \mathcal{SD}(P_k,\mathfrak{A})_\Theta$ applies (i.e., $\mathfrak{a}$ strictly dominates every other available payoff of constraint $\mathfrak{A}$ on possibility set P_k), then $\mathfrak{a} \in \mathcal{SD}_+(P,\mathfrak{A})_\Theta$ follows immediately. Consider the remaining case that $\mathfrak{a} \notin \mathcal{SD}(P_k,\mathfrak{A})_\Theta$ for every $k \in K$. Since $\mathfrak{a} \in \mathcal{SD}_+(P_k,\mathfrak{A})_\Theta$ is assumed for every $k \in K$, we obtain $\mathcal{SD}(P_k,\mathfrak{A})_\Theta = \emptyset$ and thus $\mathcal{SD}_+(P_k,\mathfrak{A})_\Theta = \mathfrak{A}$ for every $k \in K$. Consequently, our exercise is to establish $\mathcal{SD}_+(P,\mathfrak{A})_\Theta = \mathfrak{A}$. Pick some arbitrary $\mathfrak{a} \in \mathfrak{A}$ and define mapping $e_\mathfrak{a} : \mathcal{A} \times P \to \mathbb{R}$ where $e_\mathfrak{a}(\mathfrak{b},\theta) := z(\mathfrak{b},\theta) - z(\mathfrak{a},\theta)$ for every $\theta \in P$ and $\mathfrak{b} \in \mathfrak{A}$. Obviously, for every $k \in K$ there exist $\theta_k \in \Theta$ and $\mathfrak{b}_k \in \mathfrak{A}$ so that $e_\mathfrak{a}(\mathfrak{b}_k,\theta_k) \geq 0$. According to Remark B.3 net $(\theta)_{k\in K}$ contains a subnet $(\theta)_{k\in L}$ converging to some $\theta \in P$. Since $\mathfrak{A}$ is assumed to be compact, subnet $(\mathfrak{a}_k)_{k\in L}$ contains a subnet $(\mathfrak{b}_k)_{k\in M}$ converging to some $\mathfrak{a} \in \mathfrak{A}$. It follows that subnet $(\theta_k,\mathfrak{a}_k)_{k\in M}$ converges in $\Theta \times \mathfrak{A}$ to $(\theta,\mathfrak{a}) \in \Theta \times \mathfrak{A}$. To avoid a notational overload, we suppose - without loss of generality - that net $(\theta_k,\mathfrak{a}_k)_{k\in K}$ converges already to this point. By Remark B.1, mapping $e_\mathfrak{a}$ is continuous, and thus $e_\mathfrak{a}(\mathfrak{b},\theta) \geq 0$ holds. The latter means that $a_\theta \leq b_\theta$ applies. Therefore $\mathfrak{a}$ is not a payoff profile that strictly dominates on P every other available payoff profile. Because payoff profile $\mathfrak{a}$ has been arbitrarily selected, constraint $\mathfrak{A}$ does not contain a payoff profile strictly dominating every other available payoff profile. According to our specification of choice rule $\mathcal{SD}_+$, we obtain $\mathcal{SD}_+(P,\mathfrak{A})_\Theta = \mathfrak{A}$ as desired. Without difficulty, it can be demonstrated that choice rule $\mathcal{SD}_+$ is also constraint continuous from above.

• Maximin

(*Non-emptiness*) By Remark B.1, mapping z is continuous. Define mapping $z_{\inf} : \mathfrak{A} \to \mathbb{R}$ by $z_{\inf}(\mathfrak{b}) := \inf_{\theta\in P} z(\mathfrak{b},\theta)$ for every $\mathfrak{b} \in \mathfrak{A}$. Since P is assumed to be compact, standard arguments imply that $z_{\inf}$ is continuous. Since $\mathfrak{A}$ is assumed to be compact, the Weierstrass Theorem implies the existence of payoff profile $\mathfrak{a}$ satisfying $z_{\inf}(\mathfrak{a}) = \sup_{b\in\mathfrak{A}} z_{\inf}(b)$. The latter result means that there exists $\theta \in P$ so that $a_\theta \geq \mathfrak{b}_{\tilde{\theta}}$ holds for every $\tilde{\theta} \in P$ and for every $\mathfrak{b} \in \mathfrak{A}$. Hence, we have established that $\mathcal{MM}(P, \mathfrak{A})_\Theta$ is non-empty.

(*Closedness*) Let $z_{\inf} : \mathfrak{A} \to \mathbb{R}$ be the mapping introduced in the above non-emptiness proof. Define, for every $\mathfrak{a} \in \mathfrak{A}$, mapping $e_\mathfrak{a} : \mathfrak{A} \times \Theta \to \mathbb{R}$ where $e_\mathfrak{a}(\mathfrak{b}, \theta) := z_{\inf}(\mathfrak{a}) - z(\mathfrak{b}, \theta)$ holds for every $\theta \in P$ and for every $\mathfrak{b} \in \mathfrak{A}$. Since mapping z is continuous, mapping $e_\mathfrak{a} : \mathfrak{A} \times \Theta \to \mathbb{R}$ is also continuous for every $\mathfrak{a} \in \mathfrak{A}$. Note that

$$\begin{aligned}
\mathcal{MM}(P, \mathfrak{A})_\Theta &= \{\mathfrak{a} \in \mathfrak{A} : \text{ and for every } \mathfrak{b} \in \mathfrak{A}, \text{ there is some } \theta \in P \text{ so that } \inf_{\theta\in P} \mathfrak{a}_\theta \geq \mathfrak{b}_\theta.\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } \inf_{\theta\in P} \mathfrak{a}_\theta \geq \mathfrak{b}_\theta.\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \{\mathfrak{a} \in \mathfrak{A} : \text{ there is some } \theta \in P \text{ so that } e_\mathfrak{a}(\mathfrak{b}, \theta) \geq 0\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \mathrm{proj}_\mathfrak{A}^{\mathfrak{A}\times P}\{(\mathfrak{a}, \theta) \in \mathfrak{A} \times P : e_\mathfrak{a}(\mathfrak{b}, \theta) \geq 0\} \\
&= \bigcap_{\mathfrak{b}\in\mathfrak{A}} \mathfrak{A}_{e_\mathfrak{a}}^P
\end{aligned}$$

holds. Remark B.2 implies that $\mathfrak{A}_{e_\mathfrak{a}}^P$ is closed in $\mathbb{B}(\Theta)$ for every $\mathfrak{b} \in \mathfrak{A}$. Hence, the intersection of these sets is a closed subset of $\mathbb{B}(\Theta)$.

(*Continuity from above*) Consider payoff profile $\mathfrak{a} \in \mathfrak{A}$ which satisfies $\mathfrak{a} \in \cap_{k\in K}\mathcal{MM}(P_k, \mathfrak{A})_\Theta$, and let $e_\mathfrak{a}$ be the mapping introduced in the preceding proof (i.e., $e_\mathfrak{a}(\mathfrak{b}, \theta) := \inf_{\theta\in P} z(\mathfrak{a}, \theta) - z(\mathfrak{b}, \theta)$ is given for every $\theta \in P$ and for every $\mathfrak{b} \in \mathfrak{A}$). As argued there, $e_\mathfrak{a}$ is a continuous mapping. Pick some $\mathfrak{b} \in \mathfrak{A}$. Note that, for every $k \in K$ there exists $\theta_k \in P_k$ so that $e_\mathfrak{a}(\mathfrak{b}, \theta_k) \geq 0$ holds for every $\mathfrak{b} \in \mathfrak{A}$. By Remark B.3 net $(\theta_k)_{k\in K}$ contains a subnet converging to some $\theta \in P$. Without loss of generality, we assume that $(\theta_k)_{k\in K}$ converges to this point. Since $e_\mathfrak{a}$ is continuous, we obtain $e_\mathfrak{a}(\mathfrak{b}, \theta) \geq 0$. The latter result implies that $\inf_{\theta\in P} \mathfrak{a}_\theta \geq \mathfrak{b}_\theta$ is satisfied. Because payoff profile $\mathfrak{b} \in \mathfrak{A}$ has been arbitrarily selected, $\mathfrak{a} \in \mathcal{MM}(P, \mathfrak{A})_\Theta$ is established. Without difficulty, it can be demonstrated that choice rule $\mathcal{MM}$ is also constraint continuous from above.

Proof of Remark 2.4

Let $\mathfrak{B}_\epsilon(\mathfrak{a}) := \{\mathfrak{b} \in \mathbb{R}^\Omega : ||\mathfrak{b} - \mathfrak{a}||_\infty \leq \epsilon\}$ denote the ϵ-neighborhood of payoff profile $\mathfrak{a} \in \mathbb{R}^\Omega$ and radius $\epsilon > 0$ according to the sup norm $||.||_\infty$. Obviously,

$$\begin{aligned}
(\mathfrak{a}_\Theta^i)^{-1}(\mathfrak{B}_\epsilon(\mathfrak{a})) &= \left\{s^i \in S^i : z^i(s^i; .) \circ \sigma_\Theta^{-i} \in \mathfrak{B}_\epsilon(\mathfrak{a})\right\} \\
&= \left\{s^i \in S^i : ||z^i(s^i; .) \circ \sigma_\Theta^{-i} - \mathfrak{a}||_\infty \leq \epsilon\right\} \\
&= \left\{s^i \in S^i : \left|\left(z^i(s^i; .) \circ \sigma_\Theta^{-i}\right)(\theta) - \mathfrak{a}_\theta\right| \leq \epsilon \text{ for every } \theta \in \Theta\right\} \\
&= \bigcap_{\theta\in\Theta} \left\{s^i \in S^i : \left|\left(z^i(s^i; .) \circ \sigma_\Theta^{-i}\right)(\theta) - \mathfrak{a}_\theta\right| \leq \epsilon\right\} \\
&= \bigcap_{\theta\in\Theta} \left\{s^i \in S^i : \left|\left(z^i(s^i, \sigma_\Theta^{-i}(\theta))\right) - \mathfrak{a}_\theta\right| \leq \epsilon\right\} \\
&= \bigcap_{\theta\in\Theta} z^i\left(.; \sigma_\Theta^{-i}(\theta)\right)^{-1}([\mathfrak{a}_\theta - \epsilon, \mathfrak{a}_\theta + \epsilon])
\end{aligned}$$

is satisfied for every $\mathfrak{a} \in \mathbb{R}^{\Omega}$ and for every $\epsilon > 0$. Since $[\mathfrak{a}_\theta - \epsilon, \mathfrak{a}_\theta + \epsilon]$ is closed in $\mathbb{R}$ and $z^i(., \sigma_{\Theta}^{-i}(\theta))$ is continuous on S^i for every $\theta \in \Theta$, set $z^i\left(.; \sigma_{\Theta}^{-i}(\theta)\right)^{-1}([\mathfrak{a}_\theta - \epsilon, \mathfrak{a}_\theta + \epsilon])$ is closed in S^i for every $\theta \in \Theta$. Hence, the intersection of these sets is also closed in S^i. Thereby, we have established that mapping $\mathfrak{a}_{\Theta}^i$ is continuous on Θ. Because its domain is assumed to be compact and its codomain is Hausdorff, it follows from the Closed Map Lemma that mapping $\mathfrak{a}_{\Theta}^i$ is also closed. □

Proof of Remark 3.2.

(*a*) This claim can be easily proved by induction on k. (*b*) Remark (*a*) signifies that each component of deletion process $(R_k)_{k\in\mathbb{N}_0}$ is closed in S. Since R_∞ is the intersection of these components, it is closed in S too. Furthermore, since each component of this process is a restriction of Γ, it is also a restriction of Γ. It remains to show that R_∞ is non-empty. Let $\mathscr{R} := \{R_k : k \in \mathbb{N}_0\}$ be the family consisting of the components of deletion process $(R_k)_{k\in\mathbb{N}_0}$ and consider some finite subfamily $\tilde{\mathscr{R}}$ of $\mathscr{R}$. By construction, $(R_k)_{k\in\mathbb{N}_0}$ is an antitone sequence. Therefore, there exists a $R_{k^*} \in \tilde{\mathscr{R}}$ having the property that $R_{k^*} \subseteq R$ holds for any $R \in \tilde{\mathscr{R}}$. Consequently, $R_{k^*} = \cap_{R\in\tilde{\mathscr{R}}} R$ holds. By remark (*a*), R_{k^*} is non-empty and, thus, subfamily $\tilde{\mathscr{R}}$ has a non-empty intersection. Since subfamily $\tilde{\mathscr{R}}$ has been arbitrarily selected, the finite intersection property of family $\mathscr{R}$ is established. Finally, from the compactness of S it follows that $\cap_{k\in\mathbb{N}_0} R_k$ is non-empty. □

Proof of Remark 4.2

(*a*) Surjectivity follows immediately from the definition of the projection mapping. To prove continuity, consider the system

$$\mathscr{B} := \left\{ \prod_{\substack{i\in I_R \\ R\in\{S,T\}}} Q_R^i \ : \ Q_R^i \text{ is open in } R^i \right\}$$

of products of open sets. To improve the readability of the following arguments, fix $X := \Omega^{(J_S, J_T)}$ and $Y := \Omega^{(I_S, I_T)}$. Since subspace Y is supposed to be endowed with the product topology, system $\mathscr{B}$ constitutes a basis of Y. Pick some $Q \in \mathscr{B}$, i.e.,

$$Q := \prod_{\substack{i\in I_R \\ R\in\{S,T\}}} Q_R^i ,$$

where Q_R^i is open in R^i for every $R \in \{S, T\}$ and every $i \in I_R$. Because subspace X is also endowed with the product topology, the mapping $\text{proj}_{R^i}^X$ from subspace X into factor R^i is continuous for every $R \in \{S, T\}$ and every $i \in I_R$. Hence, $(\text{proj}_{R^i}^X)^{-1}(Q_R^i)$ is open in X for every $R \in \{S, T\}$ and every $i \in I_R$. Thus, the finite intersection

$$\bigcap_{\substack{i\in I_R \\ R\in\{S,T\}}} (\text{proj}_{R^i}^X)^{-1}(Q_R^i) = (\text{proj}_Y^X)^{-1}(Q)$$

is open in X, and we have shown that mapping proj_Y^X is continuous. (*b*) Since both domain and codomain of the continuous mapping proj_Y^X are compact Hausdorff, the Closed Map Lemma implies that proj_Y^X is closed. □

Proof of Remark 4.3

It is known (see for e.g. Theorem 17.16 of Aliprantis and Border, 2006) that correspondence Q^i_E is compact-valued and upper hemicontinuous if and only if for every net $(\omega^i_k, \omega^{-i}_k)_{k\in K}$ with $\omega^{-i}_k \in Q^i_E(\omega^i_k)$ for every $k \in K$ and for every limit point ω^i_* of net $(\omega^i_k)_{k\in K}$ net $(\omega^{-i}_k)_{k\in K}$ has a limit point in $Q^i_E(\omega^i_*)$. Therefore, to prove this remark, we begin by considering a net $(\omega^i_k, \omega^{-i}_k)_{k\in K}$ in Ω for which $\omega^{-i}_k \in Q^{-i}_E(\omega^i_k)$ is satisfied for every $k \in K$. Let $\omega^i_* \in \Omega^i$ be a limit point of net $(\omega^i_k)_{k\in K}$. By assumption, $(\omega^i_k, \omega^{-i}_k) \in E$ applies to every $k \in K$. Since state space Ω is compact, event E is a compact subset of Ω. For this reason, net $(\omega^i_k, \omega^{-i}_k)_{k\in K}$ has a limit point (ω^i, ω^{-i}) in E. Hence, $(\omega^i_k)_{k\in K}$ converges to ω^i and, since Ω^i is Hausdorff, this limit point is unique. Therefore, $\omega^i = \omega^i_*$ holds. It follows $\omega^{-i} \in Q^i_E(\omega^i_*)$. We see from this that $(\omega^{-i}_k)_{k\in K}$ has a limit point in $Q^i_E(\omega^i_*)$. □

Proof of Remark 4.4

First, we define correspondences $\tilde{P}^i : \Omega \to 2^{\Omega^{-i}}$ by $\tilde{P}^i := P^i \circ \mathrm{proj}_{T^i}$ and $\tilde{Q}^i_E : \Omega \to 2^{\Omega^{-i}}$ by $\tilde{Q}^i_E := Q^i_E \circ \mathrm{proj}_{\Omega^i}$. Note that since correspondence P^i is assumed to be lower hemicontinuous and mapping proj_{T^i} is continuous, correspondence $\tilde{P}^i$ is also lower hemicontinuous. Since correspondence Q^i_E is compact-valued and upper hemicontinuous (see Remark 4.3) and proj_{Ω^i} as a continuous mapping carries compact sets to compact sets, correspondence $\tilde{Q}^i$ is also compact valued and upper hemicontinuous. Without difficulty, identity

$$B^i(E) = \{\omega \in \Omega : \tilde{P}^i(\omega) \subseteq \tilde{Q}^i_E(\omega)\}$$

can be verified. In the following, we show that event $B^i(E)$ is closed. For this purpose, consider a limit point ω_* of $B^i(E)$ and some $\tilde{\omega}^{-i}_* \in \tilde{P}^i(\omega_*)$. It remains to show that $\tilde{\omega}^{-i}_*$ also belongs to $\tilde{Q}^i(\omega^*)$. By assumption, there exists a net $(\omega_k)_{k\in K}$ in $B^i(E)$ converging to ω^*. Since $\tilde{P}^i$ is lower hemicontinuous, there exists a subnet $(\omega_{k_l})_{l\in L}$ of $(\omega_k)_{k\in K}$ and a net $(\tilde{\omega}^{-i}_l)_{l\in L}$ in Ω^{-i} so that $\tilde{\omega}^{-i}_l \in \tilde{P}^i(\omega_{k_l})$ holds for every $l \in L$ and $(\tilde{\omega}^{-i}_l)_{l\in L}$ converges to $\tilde{\omega}^{-i}_*$ (see Theorem 17.19 of Aliprantis and Border, 2006). Because each member of subnet $(\omega_{k_l})_{l\in L}$ belongs to event $B^i(E)$, we obtain $\tilde{\omega}^{-i}_l \in \tilde{Q}^i_E(\omega_{k_l})$ for every $l \in L$. Since $\tilde{Q}^{-i}_E$ is compact-valued and upper hemicontinuous, subnet $(\tilde{\omega}^{-i}_l)_{l\in L}$ has a limit point in $\tilde{Q}^i_E(\omega_*)$ (see Theorem 17.16 of Aliprantis and Border, 2006). Subspace Ω^{-i} is Hausdorff and, hence, this limit point is unique. From above, we know that $\tilde{\omega}^{-i}_*$ is a limit point of $(\tilde{\omega}^{-i}_l)_{l\in L}$. Consequently, $\tilde{\omega}^{-i}_* \in \tilde{Q}^i_E(\omega_*)$ must hold. □

Proof of Remark 4.5

(*a*) We prove this claim by induction on k. The case $k = 0$ corresponds to our assumption. Suppose, for some $k \in \mathbb{N}_0$, event E_k is rectangular. Note that for every $i \in I$ event $B^i(E_k)$ is of type $\prod_{j\in I} F^j \subseteq \Omega$ where $F^j = \Omega^j$ holds for every $j \in I \setminus \{i\}$. Since for every $i \in I$ event $B^i(E_k)$ is rectangular, the intersection $\bigcap_{i\in I} B^i(E_k)$ is also rectangular. Together with our induction premise this implies that $E_{k+1} := E_k \cap \left(\bigcap_{i\in I} B^i(E_k)\right)$ is rectangular. Define $E^i_k := \mathrm{proj}_{\Omega^i}(E_k)$ for every $i \in I$ and for every $k \in \mathbb{N}_0$. We have just shown that $E_k = \prod_{i\in I} E^i_k$ holds for every $k \in \mathbb{N}_0$. Therefore, intersection

$$E_* = \bigcap_{k\in\mathbb{N}_0} E_k = \bigcap_{k\in\mathbb{N}_0} \left(\prod_{i\in I} E^i_k\right) = \prod_{i\in I} \left(\bigcap_{k\in\mathbb{N}_0} E^i_k\right)$$

is also rectangular. (b) Again, we prove this claim by induction on k. The case $k = 0$ corresponds to our assumption. Suppose, for some $k \in \mathbb{N}_0$, event E_k is closed. Due to Remark 4.4, event $B^i(E_k)$ is closed for every $i \in I$. Hence, intersection $E_{k+1} := E_k \cap \left(\bigcap_{i \in I} B^i(E_k)\right)$ is closed too. Since for any $k \in \mathbb{N}_0$ event E_k is closed, intersection $E_* := \bigcap_{k \in \mathbb{N}_0} E_k$ is closed too. $\square$

Proof of Remark 4.6

"$(i) \Rightarrow (ii)$" Set $F := E_*$. By assumption, $\omega \in F$. Consider some $\tilde{\omega} \in E_*$ and some player $i \in I$. By definition, $\tilde{\omega} \in E_{k+1}$ holds for every $k \in \mathbb{N}_0$. Hence, $\tilde{\omega} \in B^i(E_k)$ holds for every $k \in \mathbb{N}$ and, thus, $P^i(t^i_{\tilde{\omega}}) \subseteq Q^i_{E_k}(\tilde{\omega}^i)$ is satisfies for every $k \in \mathbb{N}_0$. Without difficulty, we can establish $Q^i_{E_*}(\tilde{\omega}^i) = \cap_{k \in \mathbb{N}_0} Q^i_{E_k}(\tilde{\omega}^i)$. Hence, $P^i(t^i_{\tilde{\omega}}) \subseteq Q^i_{E_*}(\tilde{\omega}^i)$ results. "$(ii) \Rightarrow (i)$" First, we show by induction that $F \subseteq E_k$ holds for any $k \in \mathbb{N}_0$. By assumption, $F \subseteq E_0$ holds. Suppose $F \subseteq E_k$ holds for some $k \in \mathbb{N}_0$ and pick some state $\tilde{\omega} \in F$. Our induction premise implies that $Q^i_F(\tilde{\omega}^i) \subseteq Q^i_{E_k}(\tilde{\omega}^i)$ is satisfied for every player $i \in I$. Consequently, $P^i(t^i_{\tilde{\omega}}) \subseteq Q^i_{E_k}(\tilde{\omega}^i)$ applies to every $i \in I$. The latter means nothing but $\tilde{\omega} \in B^i(E_k)$ for every $i \in I$. Hence, $\tilde{\omega} \in E_{k+1}$ is obtained. Since $\tilde{\omega}$ has been arbitrarily chosen, $F \subseteq \cap_{k \in \mathbb{N}_0} E_k = E_*$ is established. Because $\omega \in F$ is assumed, $\omega \in E_*$ results. "$(i) \Rightarrow (iii)$" Suppose, at state ω, event E is true and also commonly believed. Obviously, $\omega \in E_0$ holds. Pick some $k \in \mathbb{N}_0$. Because $\omega \in E_{k+1}$ is supposed, $\omega \in B^i(E_k)$ and, thus, $P^i(t^i_\omega) \subseteq Q^i_{E_k}(\omega^i)$ hold for every $i \in I$. Since k has been arbitrarily chosen, $P^i(t^i_\omega) \subseteq \cap_{k \in \mathbb{N}_0} Q^i_{E_k}(\omega) = Q^i_{E_*}(\omega)$ is satisfied. "$(iii) \Rightarrow (i)$" Let ω be a state satisfying both $\omega \in E$ and $P^i(t^i_\omega) \subseteq Q^i_{E_*}(\omega^i)$ for every player $i \in I$. By definition, $\omega \in E_0$ holds. Suppose that $\omega \in E_k$ holds for some $k \in \mathbb{N}_0$. Since $P^i(t^i_\omega) \subseteq Q^i_{E_\infty}(\omega^i) \subseteq Q^i_{E_k}(\omega^i)$ is satisfied for every $i \in I$, we obtain $\omega \in B^i(E_k)$ for every $i \in I$. Hence, $\omega \in E_k \cap \left(\bigcap_{i \in I} B^i(E_k)\right)$ applies and, thus, $\omega \in E_{k+1}$ results. Since $\omega \in E_k$ is satisfied for every $k \in \mathbb{N}_0$, we obtain $\omega \in E_*$. $\square$

Proof of Remark 4.7

We remark that $[\mathcal{C}^i]$ is a rectangular event of type $[\mathcal{C}^i] = C^i \times \Omega^{-i}$ where C^i is a subset of Ω^i. For this reason, it suffices to establish the closedness of C^i in Ω^i. Let $\omega^i_* := (s^i_*, t^i_*)$ be a limit point of C^i. Denote by $\mathscr{N}_{\omega^i_*}$ the system of all open neighborhoods of ω^i_* in Ω^i. Thus,

$$(V \setminus \{\omega^i_*\}) \cap [\mathcal{C}^i] \neq \emptyset$$

holds for any $V \in \mathscr{N}_{\omega^i_*}$. Pick, for each $V \in \mathscr{N}_{\omega^i_*}$, some $\omega^i_V := (s^i_V, t^i_V) \in \Omega^i$ for which

$$s^i_V \in \mathcal{C}^i(P^i(t^i_V), S^i)$$

is satisfied. Note that $(\omega^i_V)_{V \in \mathscr{N}_{\omega^i_*}}$ is a net directed by inclusion. We specify

$$P^i_V := \overline{\bigcup_{\substack{U \subseteq V, \\ U \in \mathscr{N}_{\omega^i_*}}} P^i(t^i_U)} \cup P^i(t^i_*)$$

for each $V \in \mathscr{N}_{\omega^i_*}$ where the upper bar about the set union is the topological closure operator. Thus, $P^i(V)$ is closed in Ω^{-i} for any $V \in \mathscr{N}_{\omega^i_*}$.

First, we establish $\bigcap_{V \in \mathscr{N}_{\omega^i_*}} P^i_V = P^i(t^i_*)$. Obviously, by construction, $\bigcap_{V \in \mathscr{N}_{\omega^i_*}} P^i_V \supseteq P^i(t^i_*)$ holds. Consider some $\omega^{-i} \in \Omega^{-i} \setminus P^i(t^i_*)$. Because Ω^{-i} is compact Hausdorff, there exists some open set Z so that $P^i(t^i_{\omega^*}) \subseteq Z$ and $\omega^{-i} \notin Z$ are satisfied. Furthermore, there exists an open set Y having the property $P^i(t^i_*) \subseteq Y \subseteq \overline{Y} \subseteq Z$. Because correspondence P^i is upper hemicontinuous, there is some open neighborhood T^i_Y of $t^i_{\omega^*}$ so that $t^i \in T^i_Y$ implies $P^i(t^i_Y) \subseteq Y$. Hence, $P^i_V \subseteq \overline{Y} \subseteq Z$ holds for any $V \in \mathscr{N}_{\omega^i_*}$ satisfying $V \subseteq S^i \times T^i_Y$. It follows that $\omega^{-i} \notin P^i_V$ is satisfied for any $V \in \mathscr{N}_{\omega^i_*}$ with $V \subseteq S^i \times T^i_Y$. Thus, $\omega^{-i} \notin \cap_{V \in \mathscr{N}_{\omega^i_*}} P^i_V$ results. Since the latter result holds for any $\omega^{-i} \in \Omega^{-i} \setminus P^i(t^i_*)$, we obtain $\bigcap_{V \in \mathscr{N}_{\omega^i_*}} P^i_V = P^i(t^i_*)$.

Next, we show that $s^i_* \in \mathcal{C}^i(P^i_V, S^i)$ holds for any $V \in \mathscr{N}_{\omega^i_*}$. Obviously, $(P^i_V)_{V \in \mathscr{N}_{\omega^i_*}}$ is an antitone net directed by inclusion. By monotonicity, we have $\mathcal{C}^i(P^i(t^i_U), S^i) \subseteq \mathcal{C}^i(P^i_U, S^i) \subseteq \mathcal{C}^i(P^i_V, S^i)$ for any $U \in \mathscr{N}_{\omega^i_*}$ with $U \subseteq V$. Therefore, $s^i_U \in \mathcal{C}^i(P^i_V, S^i)$ holds for any $U \in \mathscr{N}_{\omega^i_*}$ with $U \subseteq V$. Because net $(s^i_U)_{U \in \mathscr{N}_{\omega^i_*}, U \subseteq V}$ converges to s^i_* and choice rule $\mathcal{C}^i$ is assumed to be closed, $s^i_* \in \mathcal{C}^i(P^i_V, S^i)$ is satisfied for any $V \in \mathscr{N}_{\omega^i_*}$.

Finally, since choice rule $\mathcal{C}^i$ is continuous from above and $\bigcap_{V \in \mathscr{N}_{\omega^i_*}} P^i_V = P^i(t^i_*)$ applies,

$$\bigcap_{V \in \mathscr{N}_{\omega^i_*}} \mathcal{C}^i(P^i_V, S^i) \subseteq \mathcal{C}^i(P^i(t^i_*), S^i)$$

is true. Recall that $s^i_* \in \mathcal{C}^i(P^i_V, S^i)$ holds for any $V \in \mathscr{N}_{\omega^i_*}$ and, thus, $s^i_* \in \mathcal{C}^i(P^i(t^i_*), S^i)$ results. That means $(s^i_*, t^i_*) \in C^i$ as desired. □

Appendix C - Choice rules

The following table provides the formal specifications of all choice rules mentioned in the Introduction and in the main text of this paper.

Choice rule	Symbol	Definition
Inherent undominance	$\mathcal{IU}$	$\mathcal{IU}(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: for some non-empty $E \subseteq P$, there is no $\mathfrak{b} \in \mathfrak{A}$ so that $\mathfrak{b}_\theta \geq \mathfrak{a}_\theta$ for every $\theta \in E$ and $\mathfrak{b}_\theta > \mathfrak{a}_\theta$ for some $\theta \in E\}$
Maximin	$\mathcal{MM}$	$\mathcal{MM}(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: for every $\mathfrak{b} \in \mathfrak{A}$ there is some $\theta \in P$ so that $\inf_{\tilde{\theta} \in P} \mathfrak{a}_{\tilde{\theta}} \geq \mathfrak{b}_\theta\}$
Minimax regret	$\mathcal{MR}$	$\mathcal{MR}(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: for every $b \in \mathfrak{A}$, $\max\{c^*_\omega - a_\omega : \omega \in \Omega\} \leq \max\{c^*_\omega - b_\omega : \omega \in \Omega\}$, where $c^*_\omega := \max\{\mathfrak{c}_\omega : \mathfrak{c} \in \mathfrak{A}\}$ for every $\omega \in \Omega\}$
Point rationality	$\mathcal{PR}$	$\mathcal{PR}(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: there is some $\theta \in P$ so that $\mathfrak{a}_\theta \geq \mathfrak{b}_\theta$ for every $\mathfrak{b} \in \mathfrak{A}\}$.
Strict dominance	$\mathcal{SD}$	$\mathcal{SD}(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: $\mathfrak{a}_\theta > \mathfrak{b}_\theta$ for every $\theta \in P$ and every $\mathfrak{b} \in \mathfrak{A} \setminus \{\mathfrak{a}\}\}$.
Modified strict dominance	$\mathcal{SD}_+$	$\mathcal{SD}_+(P,\mathfrak{A})_\Theta := \begin{cases} \mathcal{SD}(P,\mathfrak{A})_\Theta & \text{if } \mathcal{SD}(P,\mathfrak{A})_\Theta \neq \emptyset, \\ \mathfrak{A} & \text{if } \mathcal{SD}(P,\mathfrak{A})_\Theta = \emptyset. \end{cases}$
Strict undominance (in pure payoff profiles)	$\mathcal{SU}_p$	$\mathcal{SU}_p(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: for every $\mathfrak{b} \in \mathfrak{A}$ there is some $\theta \in P$ so that $\mathfrak{a}_\theta \geq \mathfrak{b}_\theta\}$.
Strict undominance in mixed payoff profiles	$\mathcal{SU}_m$	$\mathcal{SU}_m(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: for every $\mu \in \Delta(\mathfrak{A})$ there is some $\theta \in P$ so that $\mathfrak{a}_\theta \geq \int_{\mathfrak{A}} \mathfrak{b}_\theta d\mu\}$.
Weak undominance (in pure payoff profiles)	$\mathcal{WU}_p$	$\mathcal{WU}_p(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: there is no $\mathfrak{b} \in \mathfrak{A}$ so that $b_\theta \geq a_\theta$ for every $\theta \in P$ and $b_\theta > a_\theta$ for some $\theta \in P\}$
Weak undominance in mixed payoff profiles	$\mathcal{WU}_m$	$\mathcal{WU}_m(P,\mathfrak{A})_\Theta := \{\mathfrak{a} \in \mathfrak{A}$: there is no probability measure $\mu \in \Delta(\mathfrak{A})$ so that $\int_{\mathfrak{A}} \mathfrak{b}_\theta d\mu \geq \mathfrak{a}_\theta$ for every $\theta \in P$ and $\int_{\mathfrak{A}} \mathfrak{b}_\theta d\mu > \mathfrak{a}_\theta$ for some $\theta \in P\}$

Table C.1: Specifications of choice rules